# Division by Zero Calculus in Figures
# – Our New Space Since Euclid –
# (Draft)

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**Abstract:** We will show in this book that our basic idea for our space is wrong since Euclid, simply and clearly for some general people over high school students by using many simple figures. The common sense on the division by zero with a long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on derivatives we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero. In this book, we will show and give various applications of the division by zero $0/0 = 1/0 = z/0 = 0$ with many figures. In particular, we will introduce several fundamental concepts on Euclidian geometry which show new elementary concepts on our space. We will know that the division by zero is our elementary and fundamental mathematics.

**Key Words:** Division by zero, division by zero calculus, singularity, derivative, $0/0 = 1/0 = z/0 = 0$, $\tan(\pi/2) = 0$, infinity, discontinuous, point at infinity, gradient, Laurent expansion, Euclidean geometry, Wasan.

**Preface**

The division by zero has a long and mysterious story over the world (see, for example, H. G. Romig [19] and Google site with the division by zero) with its physical viewpoints since the document of zero in India on AD 628. In particular, note that Brahmagupta (598 -668 ?) established the four arithmetic operations by introducing 0 and at the same time he defined as $0/0 = 0$ in Brhmasphuasiddhnta. Our world history, however, stated that his definition $0/0 = 0$ is wrong over 1300 years, but, we will see that his definition is right and suitable.

The division by zero $1/0 = 0/0 = z/0$ itself will be quite clear and trivial with several natural extensions of the fractions against the mysterously long history, as we can see from the concepts of the Moore-Penrose generalized inverses or the Tikhonov regularization method to the fundamental equation $az = b$, whose solution leads to the definition $z = b/a$.

However, the result (definition) will show that for the elementary mapping

$$W = \frac{1}{z}, \tag{0.1}$$

the image of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere ([2]). As the representation of the point at infinity of the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and $\infty$ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value by the limiting like $+\infty$ and $-\infty$ or the point at infinity as $\infty$. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotle. – For the related Greece philosophy, see [29, 30, 31]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([11, 14]) for example. Therefore, the division by zero will give great impacts to calculus, Euclidean geometry, analytic geometry, differential equations, complex analysis in the undergraduate level and to our basic ideas for the space and universe.

We have to arrange globally our modern mathematics in our undergraduate level. Our common sense on the division by zero will be wrong, with our basic idea on the space and the universe since Aristotle and Euclid. We would like to show clearly these facts in this book, in particular, with elementary geometry for some general people. The purpose of this book is to show our new space concepts clearly and for this purpose we will use many simple figures.

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Meanwhile, we are having interesting negative comments from several people on our division by zero. However, they seem to be just traditional and old feelings, and they are not reasonable at all for the authors. The typical good comment for the first draft is given by some physician as follows:

*Here is how I see the problem with prohibition on division by zero, which is the biggest scandal in modern mathematics as you rightly pointed out* (2017.10.14.08:55).

**Contents**

# 1 Introduction - Simple History of the Division by Zero

By a **natural extension** of the fractions

$$\frac{b}{a} \tag{1.1}$$

for any complex numbers $a$ and $b$, we found the simple and beautiful result, for any complex number $b$

$$\frac{b}{0} = 0, \tag{1.2}$$

incidentally in [21] by the Tikhonov regularization for the Hadamard product inversions for matrices and we discussed their properties and gave several physical interpretations on the general fractions in [6] for the case of real numbers. The result is a very special case for general fractional functions in [4].

The division by zero has a long and mysterious story over the world (see, for example, H. G. Romig [19] and Google site with the division by zero) with its physical viewpoints since the document of zero in India on AD 628. In particular, note that Brahmagupta (598 -668 ?) established the four arithmetic operations by introducing 0 and at the same time he defined as $0/0 = 0$ in Brhmasphuasiddhnta. Our world history, however, stated that his definition $0/0 = 0$ is wrong over 1300 years, but, we will see that his definition is right and suitable.

Indeed, we will show typical examples for $0/0 = 0$. However, in this introduction, these examples are based on some natural feelings and are not mathematics, because we do still not give the definition of $0/0$. However, following our new mathematics, these examples and results may be accepted as natural ones later:

The conditional probability $P(A|B)$ for the probability of $A$ under the condition that $B$ happens is given by the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If $P(B) = 0$, then, of course, $P(A \cap B) = 0$ and from the meaning, $P(A|B) = 0$ and so, $0/0 = 0$.

For the representation of inner product in vectors

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB}$$

$$= \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2+A_y^2+A_z^2}\sqrt{B_x^2+B_y^2+B_z^2}},$$

if $\mathbf{A}$ or $\mathbf{B}$ is the zero vector, then we see that $0 = 0/0$. In general, the zero vector is orthogonal for any vector and then, $\cos\theta = 0$.

For the differential equation

$$\frac{dy}{dx} = \frac{2y}{x},$$

we have the general solution with constant $C$

$$y = Cx^2.$$

At the origin $(0,0)$ we have

$$y'(0) = \frac{0}{0} = 0.$$

For three points $a, b, c$ on a circle with center at the origin on the complex $z$-plane with radius $R$, we have

$$|a+b+c| = \frac{|ab+bc+ca|}{R}.$$

If $R = 0$, then $a, b, c = 0$ and we have $0 = 0/0$.

For a circle with radius $R$ and for an inscribed triangle with side lengths $a, b, c$, and further for the inscribed circle with radius $r$ for the triangle, the area $S$ of the triangle is given by

$$S = \frac{r}{2}(a+b+c) = \frac{abc}{4R}. \tag{1.3}$$

If $R = 0$, then we have

$$S = 0 = \frac{0}{0} \tag{1.4}$$

(H. Michiwaki: 2017.7.28.).

We have furthermore many and concrete examples as we will see in this book.

However, we do not know the reason and motivation of the definition of $0/0 = 0$ by Brahmagupta, furthermore, for the important case $1/0$ we do not know any result there. – Indeed, we find many and many wrong logics on the division by zero, without the definition of the division by zero $z/0$. However, Sin-Ei Takahasi ([6]) discovered a simple and decisive interpretation (1.2) by analyzing the extensions of fractions and by showing the complete characterization for the property (1.2):

**Proposition 1.** *Let $F$ be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ satisfying*

$$F(b, a)F(c, d) = F(bc, ad)$$

*for all*

$$a, b, c, d \in \mathbf{C}$$

*and*

$$F(b, a) = \frac{b}{a}, \quad a, b \in \mathbf{C}, a \neq 0.$$

*Then, we obtain, for any $b \in \mathbf{C}$*

$$F(b, 0) = 0.$$

Note that the complete proof of this proposition is simply given by 2 or 3 lines, as we will give its complete proof later.

In a long mysterious history of the division by zero, this proposition seems to be decisive. Since the publication of the paper, over fully four years we see still curious information on the division by zero and we see still many wrong opinions on the division by zero.

Indeed, the Takahasi's assumption for the product property should be accepted for any generalization of fraction (division). Without the product property, we will not be able to consider any reasonable fraction (division).

Following the proposition, we should **define**

$$F(b, 0) = \frac{b}{0} = 0,$$

and consider, for any complex number $b$, as (1.2); that is, for the mapping

$$W = \frac{1}{z}, \tag{1.5}$$

the image of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere ([2]). As the representation of the point at infinity of the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and $\infty$ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value by the limiting like $+\infty$ and $-\infty$ or the point at infinity as $\infty$. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotle. – For the related Greece philosophy, see [29, 30, 31]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([11, 14]) for example. Therefore, the division by zero will give great impacts to calculus, Euclidian geometry, analytic geometry, complex analysis and the theory of differential equations in an undergraduate level and furthermore to our basic ideas for the space and universe.

Meanwhile, the division by zero (1.2) was derived from several independent approaches as in:

1) by the generalization of the fractions by the Tikhonov regularization or by the Moore-Penrose generalized inverse to the fundamental equation $az = b$ that leads to the definition of the fraction $z = b/a$,

2) by the intuitive meaning of the fractions (division) by H. Michiwaki,

3) by the unique extension of the fractions by S. Takahasi, as in the above,

4) by the extension of the fundamental function $W = 1/z$ from $\mathbf{C} \setminus \{0\}$ into $\mathbf{C}$ such that $W = 1/z$ is a one to one and onto mapping from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C} \setminus \{0\}$ and the division by zero $1/0 = 0$ is a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$, – Here, we can consider the above on the real numbers $\mathbf{R}$ for the function $y = 1/x$ –

and

5) by considering the values of functions with the mean values of functions.

Furthermore, in ([10]) we gave the results in order to show the reality of the division by zero in our world:

A) a field structure containing the division by zero — the **Yamada field Y**,

B) by the gradient of the $y$ axis on the $(x, y)$ plane — $\tan \frac{\pi}{2} = 0$,

C) by the reflection $W = 1/\overline{z}$ of $W = z$ with respect to the unit circle with center at the origin on the complex $z$ plane — the reflection point of zero is zero, (The classical result is wrong, see [14]),

and

D) by considering rotation of a right circular cone having some very interesting phenomenon from some practical and physical problem.

Furthermore, in ([11],[21]), we discussed many division by zero properties in the Euclidean plane - however, precisely, our new space is not the Euclidean space. More recently, we see the great impact to Euclidian geometry in connection with Wasan in ([15, 16, 17]). In ([7]), we gave beautiful geometrical interpretations of determinants from the viewpoint of the division by zero.

In this book, in order to show simply our new space introduced by the division by zero we will discuss the division by zero in Euclidian geometry by using many simple figures. We will be able to see that the division by zero is our elementary and fundamental mathematics.

# 2 Introduction and definitions of general fractions

We first introduce several definitions of our general fractions containing the division by zero. We will give the logical background simply and essential principles for our division by zero.

## 2.1 By the Tikhonov regularization

For any real numbers $a$ and $b$ containing 0, we will introduce general fractions

$$\frac{b}{a}. \tag{2.1}$$

We will think that for the fraction (2.1), it will be given by the solution of the equation

$$ax = b.$$

Here, in order to see its essence, we will consider all on the real number field $\mathbf{R}$. However, since $0 \cdot x = 0$, for $b \neq 0$, this equation has not any solution for the case $a = 0$, and so, by the concept of the Tikhonov regularization method, we will consider the equation as follows:

For any fixed $\lambda > 0$, the minimum member of the Tikhonov function in $x$

$$\lambda x^2 + (ax - b)^2; \tag{2.2}$$

that is,

$$x_\lambda(a,b) = \frac{ab}{\lambda + a^2} \tag{2.3}$$

may be considered as the fraction in the sense of Tikhonov with parameter $\lambda$, in a generalized sense. Note that the limit

$$\lim_{\lambda \to +0} x_\lambda(a,b)$$

exists always. By the limit

$$\lim_{\lambda \to +0} x_\lambda(a,b) = \frac{b}{a}, \tag{2.4}$$

we will define the general fractions $\frac{b}{a}$.

Note that, for $a \neq 0$, the definition (2.4) is the same as the ordinary sense, however, when $a = 0$, we obtain the desired results $b/0 = 0$, since $x_\lambda(0, b) = 0$, always.

The result (2.4) is a trivial **Moore-Penrose generalized inverse** (solution) for the equation $ax = b$ that is well-established and fundamental. For this reason, we may consider that the division by zero is trivial and clear against the great and mysterious history of the division by zero.

For the general theory of the Tikhonov regularization and many applications, see the cited references, for example, [22].

## 2.2 By the Takahasi uniqueness theorem

Sin-Ei, Takahashi ([25]) established a simple and natural interpretation (1.2) by analyzing any extensions of fractions and by showing the complete characterization for such property (1.2). Furthermore, he examined several fundamental properties of the general fractions. His result will show that our mathematics says that the results (1.2) should be accepted as natural ones.

**Theorem** *Let $F$ be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ such that*

$$F(a, b)F(c, d) = F(ac, bd)$$

*for all*

$$a, b, c, d \in \mathbf{C}$$

*and*

$$F(a, b) = \frac{a}{b}, \quad a, b \in \mathbf{C}, b \neq 0.$$

*Then, we obtain, for any $a \in \mathbf{C}$*

$$F(a, 0) = 0.$$

*Proof.* We have $F(a, 0) = F(a, 0)1 = F(a, 0)\frac{2}{2} = F(a, 0)F(2, 2) = F(a \cdot 2, 0 \cdot 2) = F(2a, 0) = F(2, 1)F(a, 0) = 2F(a, 0)$.

Thus $F(a, 0) = 2F(a, 0)$ which implies the desired result $F(a, 0) = 0$ for all $a \in \mathbf{C}$.

Several mathematicians pointed out to the authors for the publication of the paper ([6]) that the notations of 100/0 and 0/0 are not good for the sake of the generalized senses, however, there does not exist other natural and good meaning for them. Why should we need and use any new notations for involving the notations? We should use the notation, we think so. Indeed, we will see in this book that many and many fractions in our formulas will have this meaning with the concept of the division by zero calculus for the case of functions.

Note, in particular, that the Takahasi assumption on the product property for fractions (division) is fundamental and if the assumption is not satisfied, any fractions (division) in a generalized sense will not have a reasonable sense or will not be a good existence.

## 2.3 By the intuitive meaning of the fractions (division) by H. Michiwaki

We will introduce an another approach. The division $b/a$ may be defined **independently of the product**. Indeed, in Japan, the division $b/a$ ; $b$ **raru** $a$ (**jozan**) is defined as how many $a$ exists in $b$, this idea comes from subtraction $a$ repeatedly. (Meanwhile, product comes from addition). In Japanese language for "division", there exists such a concept independently of product. H. Michiwaki and his 6 years old daughter said for the result $100/0 = 0$ that the result is clear, from the meaning of the fractions independently of the concept of product and they said: $100/0 = 0$ does not mean that $100 = 0 \times 0$. Meanwhile, many mathematicians had a confusion for the result. Her understanding is reasonable and may be acceptable: $100/2 = 50$ will mean that we divide 100 by 2, then each will have 50. $100/10 = 10$ will mean that we divide 100 by 10, then each will have 10. $100/0 = 0$ will mean that we do not divide 100, and then nobody will have at all and so 0. Furthermore, she said then the rest is 100; that is, mathematically;

$$100 = 0 \cdot 0 + 100.$$

Now, all the mathematicians may accept the division by zero $100/0 = 0$ with natural feelings as a trivial one?

For simplicity, we shall consider the numbers on non-negative real numbers. We wish to define the division (or fraction) $b/a$ following the usual procedure for its calculation, however, we have to take care for the division by zero: The first principle, for example, for 100/2 we shall consider it as follows:

$$100 - 2 - 2 - 2-, ..., -2.$$

How may times can we subtract 2? At this case, it is 50 times and so, the fraction is 50. The second case, for example, for $3/2$ we shall consider it as follows:

$$3 - 2 = 1$$

and the rest (remainder) is 1, and for the rest 1, we multiple 10, then we consider similarly as follows:

$$10 - 2 - 2 - 2 - 2 - 2 = 0.$$

Therefore $10/2 = 5$ and so we define as follows:

$$\frac{3}{2} = 1 + 0.5 = 1.5.$$

By these procedures, for $a \neq 0$ we can define the fraction $b/a$, usually. Here we do not need the concept of product. Except the zero division, all the results for fractions are valid and accepted. Now, we shall consider the zero division, for example, $100/0$. Since

$$100 - 0 = 100,$$

that is, by the subtraction $100 - 0$, 100 does not decrease, so we can not say we subtract any from 100. Therefore, the subtract number should be understood as zero; that is,

$$\frac{100}{0} = 0.$$

We can understand this: the division by 0 means that it does not divide 100 and so, the result is 0. Similarly, we can see that

$$\frac{0}{0} = 0.$$

As a conclusion, we should define the zero division as, for any $b$

$$\frac{b}{0} = 0.$$

See [6] for the details.

## 2.4 Other introductions of general fractions

By the extension of the fundamental function $W = 1/z$ from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C}$ such that $W = 1/z$ is a one to one and onto mapping from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C} \setminus \{0\}$ and the division by zero $1/0 = 0$ is a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$.

In this paragraph, we can consider in the above, for complex, real, for $W$, $y$, for $z$, $x$ and for $\mathbf{C}$, $\mathbf{R}$.

On the division by zero in our theory, we will need only one new assumption in our mathematics that for the elementary function $W = 1/z$, $W(0) = 0$. However, for algebraic calculation of the division by zero, we have to follow the law of the **Yamada field**. As the number system containing the division by zero, the concept of the Yamada field is very fundamental, however, for some simple book for some general people, we do not refer to it. For functions, however, we have to consider the concept of **the division by zero calculus**.

# 3 Division by zero calculus

As the number system containing the division by zero, the Yamada field structure is complete.

However for applications of the division by zero to **functions**, we will need the concept of division by zero calculus for the sake of uniquely determinations of the results and for other reasons. See [11].

For example, for the typical linear mapping

$$y = \frac{x-1}{x+1}, \tag{3.1}$$

it gives a mapping on $\{\mathbf{R} \setminus \{-1\}\}$ onto $\{\mathbf{R} \setminus \{1\}\}$ in one to one and from

$$y = 1 + \frac{-2}{x-(-1)}, \tag{3.2}$$

we see that $-1$ corresponds to $1$ and so the function maps the whole $\{\mathbf{R}\}$ onto $\{\mathbf{R}\}$ in one to one.

Meanwhile, note that for

$$y = (x-1) \cdot \frac{1}{x+1}, \tag{3.3}$$

we should not enter $x = -1$ in the way

$$[(x-1)]_{x=1} \cdot \left[ \frac{1}{x+1} \right|_{x=1} = (-2) \cdot 0 = 0. \tag{3.4}$$

However, in may cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We will refer to this delicate problem with many examples.

## 3.1 Introduction of the division by zero calculus

We will introduce the division by zero calculus: For any formal (Laurent) expansion around $x = a$,

$$f(x) = \sum_{n=-\infty}^{-1} C_n (x-a)^n + C_0 + \sum_{n=1}^{\infty} C_n (x-a)^n \tag{3.5}$$

we obtain the identity, by the division by zero

$$f(a) = C_0. \tag{3.6}$$

Note that here, there is no problem on any convergence of the expansion (3.5) at the point $x = a$.

For the correspondence (3.6) for the function $f(x)$, we will call it **the division by zero calculus**. By considering the formal derivatives in (3.5), we can define any order derivatives of the function $f$ at the singular point $a$ as follows:

$$f^{(n)}(a) = n!C_n.$$

In order to avoid any logical confusion in the division by zero, we would like to refer to the logical essence:

**For the elementary function $W = f(z) = 1/z$, we define $f(0) = 0$ and we will write it by $1/0$ following the form, apart from the sense of the intuitive sense of fraction. With only this new definition, we can develop our mathematics, through the division by zero calculus.**

As a logical line for the division by zero, we can consider as follows:

We define $1/0 = 0$ for the form; this precise meaning is that for the function $W = f(x) = 1/x$, we have $f(0) = 0$ following the form. Then, we can define the division by zero calculus (3.6) for (3.5). In particular, from the function $f(x) \equiv 0$ we have $0/0 = 0$. In this sense, $1/0 = 0$ is more fundamental than $0/0 = 0$; that is, from $1/0 = 0$, $0/0 = 0$ is derived.

For some general people, we would like to refer to some simple facts:

The Laurent expansion (3.5) is considered in the theory of **analytic function theory**, but we can consider it a formal expansion of the function $f(x)$ around the point at $a$ and the summation may be considered as in a finite summation. The coefficients $C_n$ are determined uniquely as in the coefficients in polynomials. The Laurent coefficients are determined formally and for typical functions we can find the Laurent expansions by many hand books; that is, the division by zero calculus may be calculated formally.

However, for functions we see that the results by the division by zero calculus have not always practical senses and so, **for the results by division by zero we should check the results, case by case.**

For example, for the simple example for the line equation on the $x, y$ plane

$$ax + by + c = 0$$

we have, formally

$$x + \frac{by + c}{a} = 0,$$

and so, by the division by zero, we have, for $a = 0$, the reasonable result

$$x = 0.$$

Indeed, for the equation $y = mx$, from

$$\frac{y}{m} = x,$$

we have, by the division by zero, $x = 0$ for $m = 0$. This gives the case $m = \pm\infty$ of the gradient of the line. – This will mean that the equation $y = mx$ represents the general line through the origin in this sense. – This method was applied in many cases, for example see [15, 16].

However, from

$$\frac{ax + by}{c} + 1 = 0,$$

for $c = 0$, we have the contradiction, by the division by zero

$$1 = 0.$$

Meanwhile, note that for the function $f(x) = x + \frac{1}{z}$, $f(0) = 0$, however, for the function

$$f(x)^2 = x^2 + 2 + \frac{1}{x^2},$$

we have $f^2(0) = 2$. Of course,

$$f(0) \cdot f(0) = \{f(0)\}^2 = 0.$$

Furthermore, see many examples, [11].

## 3.2 Ratio

On the real $x-$ line, we fix two different point $P_1(x_1)$ and $P_2(x_2)$ and we will consider the point, with a real number

$$P(x;r) = \frac{x_1 + rx_2}{1 + r}. \tag{3.7}$$

If $r = 1$, then the point $P(x;1)$ is the mid point of the two points $P_1$ and $P_2$ and for $r > 0$, the point $P$ is on the interval $(x_1, x_2)$. Meanwhile, for $-1 < r < 0$, the point $P$ is on $(-\infty, x_1)$ and for $r < -1$, the point $P$ is on $(x_2, +\infty)$. Of course, for $r = 0$, $P = P_1$. We see that $r$ tends to $+\infty$ and $-\infty$, $P$ tends to the point $P_2$. We see the pleasant fact that by the division by zero calculus, $P(x, -1) = P_2$. For this fact we see that for all real numbers $r$ correspond to all real line numbers.

In particular, we see that in many text books on the undergraduate course the formula (3.7) is stated as a parameter representation of the line through the two pints $P_1$ and $P_2$. However, if we do not consider the case $r = -1$ by the division by zero calculus, the classical statement is not right, because the point $P_2$ may not be considered.

On this setting, we will consider another representation

$$P(x;m,n) = \frac{mx_2 - nx_1}{m - n}$$

for the exterior division point $P(x;m,n)$ in $m:n$ for the point $P_1$ and $P_2$. For $m = n$. we obtain, by the division by zero calculus, $P(x;m,m) = x_2$. Imagine the result that the point $P(x;m,m) = P_2$ and the point $P_2$ seems to be the point $P_1$. Such a strong discontinuity happens for many cases. See [11, 14].

By the division by zero, we can introduce the ratio for any complex numbers $a, b, c, d$ as

$$\frac{AC}{CB} = \frac{c - a}{b - c}. \tag{3.8}$$

We will consider the **Appollonius circle** determined by the equation

$$\frac{AP}{PB} = \frac{|z - a|}{|b - z|} = \frac{m}{n} \tag{3.9}$$

for fixed $m, n \geq 0$. Then, we obtain the equation for the cirlce

$$\left| z - \frac{-n^2 a + m^2 b}{m^2 - n^2} \right|^2 = \frac{m^2 n^2}{(m^2 - n^2)^2} \cdot |b - a|^2. \tag{3.10}$$

If $m = n \neq 0$, the circle is the line in (3.10). For $|m| + |n| \neq 0$, if $m = 0$, then $z = a$ and if $n = 0$, then $z = b$. If $m = n = 0$ then $z$ is $a$ or $b$.

The representation (3.10) is valid always, however, (3.10) is not reasonable for $m = n$. The property of the division by zero depends on the representations of formulas.

On the real line, the points $P(p), Q(1), R(r), S(-1)$ form a harmonic range of points if and only if

$$p = \frac{1}{r}.$$

If $r = 0$, then we have $p = 0$ that is now the representation of the point at infinity(see Figure 1) (H. Okumura: 2017.12.27.).

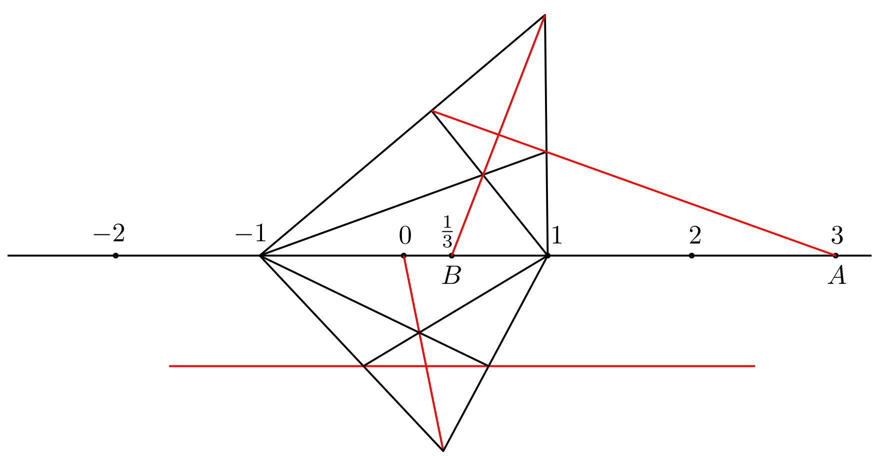


Figure 1.

## 3.3 Remarks for the applications of the division by zero and the division by zero calculus

As the number system, we can calculus by the Yamada field structure. However, for functions, the problems are involved for their structures and we have also the delicate problems for the smoothness of functions. So, by applying the division by zero, we should consider and apply the division by zero and division by zero calculus in many ways and **check the results** obtained. By considering many ways, we will be able to see many new aspects and results. By checking the results obtained, we will be able to find new prospects. With this idea, we can enjoy the division by zero calculus **with free spirits without logical problems.** – In this idea, we may ask what is mathematics?

# 4 Derivatives of a function

On derivatives, we obtain new concepts, from the division by zero. We will consider the fundamentals, first.

From the viewpoint of the division by zero, when there exists the limit, at $x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \infty \tag{4.1}$$

or

$$f'(x) = -\infty, \tag{4.2}$$

both cases, we can write them as follows:

$$f'(x) = 0. \tag{4.3}$$

This property was derived from the fact that the gradient of the $y$ axis is zero; that is,

$$\tan \frac{\pi}{2} = 0, \tag{4.4}$$

that was derived from many geometric properties in [11], and also in the formal way from the result $1/0 = 0$. Of course, by the division by zero calculus, we can derive the result.

For the double angle formula

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}, \tag{4.5}$$

for $\alpha = \pi/2$, we have:

$$0 = \frac{2 \cdot 0}{1 - 0}. \tag{4.6}$$

We will look this fundamental result by elementary functions. For the function

$$y = \sqrt{1 - x^2}, \tag{4.7}$$

$$y' = \frac{-x}{\sqrt{1 - x^2}}, \tag{4.8}$$

and so,

$$[y']_{x=1} = 0, \quad [y']_{x=-1} = 0. \tag{4.9}$$

Of course, depending on the context, we should refer to the derivatives of a function at a point from the right hand direction and the left hand direction.

Here, note that, for $x = \cos\theta, y = \sin\theta$,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\left(\frac{dx}{d\theta}\right)^{-1} = -\cot\theta.$$

Note also that from the expansion

$$\cot z = \frac{1}{z} + \sum_{\nu=-\infty,\nu\neq 0}^{+\infty}\left(\frac{1}{z-\nu\pi} + \frac{1}{\nu\pi}\right) \tag{4.10}$$

or the Laurent expansion

$$\cot z = \sum_{n=-\infty}^{\infty}\frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} z^{2n-1},$$

we have

$$\cot 0 = 0.$$

Note that in (4.10), since

$$\left(\frac{1}{z-\nu\pi} + \frac{1}{\nu\pi}\right)_{\nu=0} = \frac{1}{z}, \tag{4.11}$$

we can write it simply

$$\cot z = \sum_{\nu=-\infty}^{+\infty}\left(\frac{1}{z-\nu\pi} + \frac{1}{\nu\pi}\right). \tag{4.12}$$

The differential equation

$$y' = -\frac{x}{y} \tag{4.13}$$

with a general solution

$$x^2 + y^2 = a^2 \tag{4.14}$$

is satisfied for all the points of the solutions by the division by zero, however, the differential equations

$$x + yy' = 0, \quad y' \cdot \frac{y}{x} = -1 \tag{4.15}$$

are not satisfied for the all points of the solutions, because they may not be considered at the points $(0, -a)$ and $(0, a)$ in the usual sense.

For the function $y = \log x$,

$$y' = \frac{1}{x}, \tag{4.16}$$

and so,

$$[y']_{x=0} = 0. \tag{4.17}$$

For the elementary ordinary differential equation

$$y' = \frac{dy}{dx} = \frac{1}{x}, \quad x > 0, \tag{4.18}$$

how will be the case at the point $x = 0$? From its general solution, with a general constant $C$ (see Figure 2)

$$y = \log x + C, \tag{4.19}$$

we see that

$$y'(0) = \left[\frac{1}{x}\right]_{x=0} = 0, \tag{4.20}$$

that will mean that the division by zero $1/0 = 0$ is very natural.

In addition, note that the function $y = \log x$ has infinite order derivatives and all the values are zero at the origin, in the sense of the division by zero.

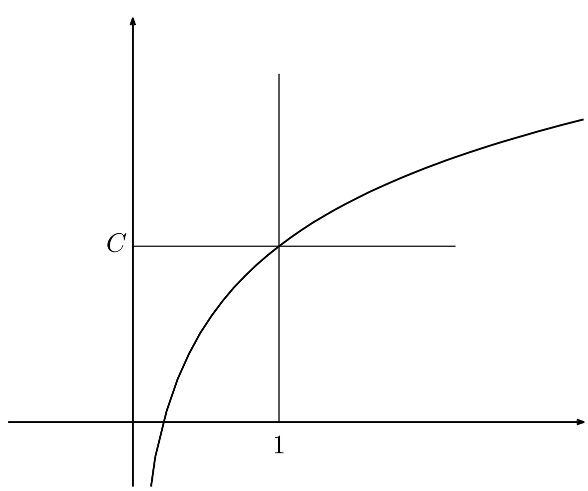


Figure 2.

However, for the derivative of the function $y = \log x$, we have to fix the sense at the origin, clearly, because the function is not differentiable, but it has a singularity at the origin. For $x > 0$, there is no problem for (4.16) and (4.17). At $x = 0$, we see that we can not consider the limit in the sense (4.1). However, $x > 0$ we have (4.17) and

$$\lim_{x \to +0} (\log x)' = +\infty. \tag{4.21}$$

In the usual sense, the limit is $+\infty$, but in the present case, in the sense of the division by zero, we have:

$$\left[(\log x)'\right]_{x=0} = 0 \tag{4.22}$$

and we will be able to understand its sense graphically.

# 5 Triangles and division by zero

In order to see how elementary of the division by zero, we will see the division by zero in triangles as the fundamental objects. Even the case of triangles, we can derive new concepts and results.

We will consider a triangle ABC with length $a, b, c$. Let $\theta$ be the angle of the side BC and the bisector line of A. Then, we have the identity

$$\tan\theta = \frac{c+b}{c-b}\tan\frac{A}{2}, \quad b < c.$$

For $c = b$, we have

$$\tan\theta = \frac{2b}{0}\tan\frac{A}{2}.$$

Of course, $\theta = \pi/2$; that is,

$$\tan\frac{\pi}{2} = 0.$$

Here, we used

$$\frac{2b}{0} = 0$$

and not by the division by zero calculus

$$\frac{c+b}{c-b} = 1 + \frac{2b}{c-b}$$

and for $c = b$

$$\frac{c+b}{c-b} = 1.$$

Of course, $\theta = \pi/2$.

We have the formula

$$\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{\tan B}{\tan C}.$$

If $a^2+b^2-c^2 = 0$, then $C = \pi/2$. Then,

$$0 = \frac{\tan B}{\tan\frac{\pi}{2}} = \frac{\tan B}{0}.$$

Meanwhile, for the case $a^2-b^2+c^2 = 0$, then $B = \pi/2$, and we have

$$\frac{a^2+b^2-c^2}{0} = \frac{\tan\frac{\pi}{2}}{\tan C} = 0.$$

Meanwhile, the lengths $f$ and $f'$ of the bisector lines of A and in the out of the triangle ABC are given by

$$f = \frac{2bc \cos \frac{A}{2}}{b+c}$$

and

$$f' = \frac{2bc \sin \frac{A}{2}}{b-c},$$

respectively.

If $b = c$, then we have $f' = 0$, by the division by zero. However, note that, from

$$f' = 2 \sin \frac{A}{2} \left( c + \frac{c^2}{b-c} \right),$$

by the division by zero calculus, for $b = c$, we have

$$f' = 2b \sin \frac{A}{2} = a.$$

The result $f' = 0$ is a now popular property, but the result $f' = a$ is also an interesting popular property. See [11].

Let H be the perpendicular leg of A to the side BC and let E and M be the mid points of AH and BC, respectively. Let $\theta$ be the angle of EMB $(b > c)$. Then, we have

$$\frac{1}{\tan \theta} = \frac{1}{\tan C} - \frac{1}{\tan B}.$$

If $B = C$, then $\theta = \pi/2$ and $\tan(\pi/2) = 0$.

In the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

if $b$ or $c$ is zero, then, by the division by zero, we have $\cos A = 0$. Therefore, then we should understand as $A = \pi/2$.

This result may be derived from the formulas

$$\sin^2 \frac{A}{2} = \frac{(a-b+c)(a+b-c)}{4bc}$$

and

$$\cos^2 \frac{A}{2} = \frac{(a+b+c)(-a+b+c)}{4bc},$$

by applying the division by zero calculus.

Let $r$ be the radius of the inscribed circle of the triangle ABC, and $r_A, r_B, r_C$ be the distances from A,B,C to the lines BC, CA, AB, respectively. Then we have

$$\frac{1}{r} = \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C}.$$

When the point A is the point at infinity, then, $r_A = 0$ and $r_B = r_C = 2r$ and the identity still holds.

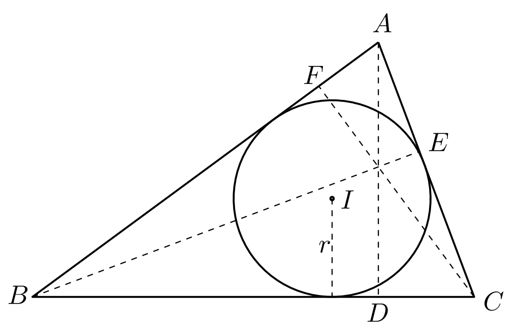


Figure 3.

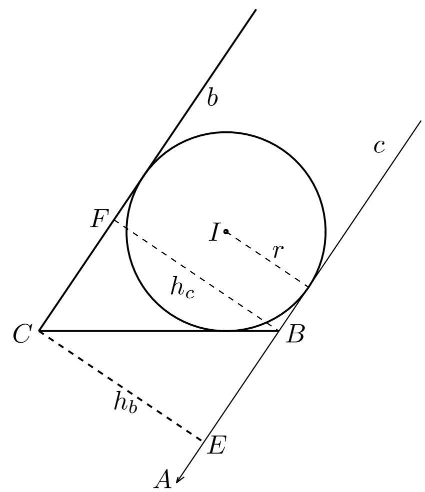


Figure 4.

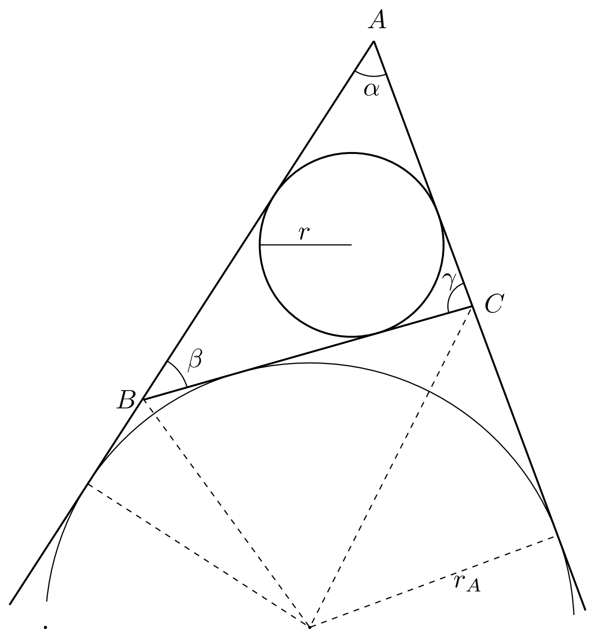


Figure 5.

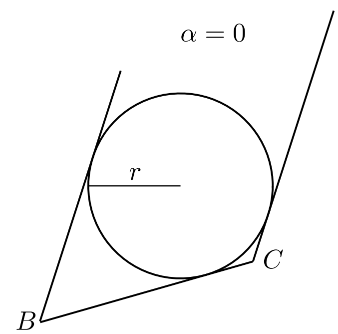


Figure 6.

We have the identities, for the radius $R$ of the circumscribed circle of the triangle ABC,

$$S = \frac{ar_A}{2} = \frac{1}{2}bc\sin A$$
$$= \frac{1}{2}a^2\frac{\sin B\sin C}{\sin A}$$
$$= \frac{abc}{4R} = 2R^2\sin A\sin B\sin C = rs, \quad s = \frac{1}{2}(a+b+c).$$

If A is the point at infinity, then, $S = s = r_A = b = c = 0$ and the above identities all valid.

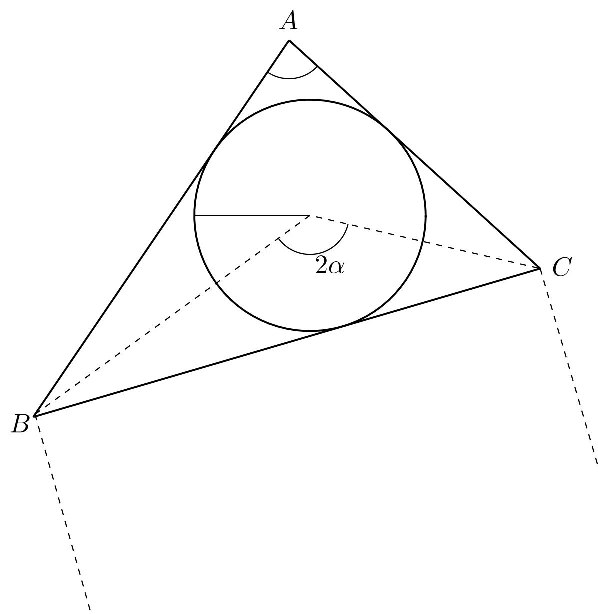


Figure 7.

For the identity

$$\tan\frac{A}{2} = \frac{r}{s-a},$$

if the point A is the point at infinity, $A = 0, s-a = 0$ and the identity holds as $0 = r/0$. Meanwhile, if $A = \pi$, then the identity holds as $\tan(\pi/2) = 0 = 0/s$.

In a triangle ABC, let X be the leg of the perpendicular line from A to the line BC and let Y be the common point of the bisector line of A and the line BC. Let P and Q be the tangential points on the line BC with the incircle of the triangle and the escribed circle in the sector with the angle A, respectively. Then, we know that

$$\frac{XP}{PY} = \frac{XQ}{QY}.$$

If $AB = AC$, then, of course, X=Y=P=Q. Then, we have:

$$\frac{0}{0} = \frac{0}{0} = 0.$$

Let X,Y, Q be the common points with a line and three lines AC, BC and AB, respectively. Let P be the common point with the line AB and the line through the point C and the common point of the lines AY and BX. Then, we know the identity

$$\frac{AP}{AQ} = \frac{BP}{BQ}.$$

If two lines XY and AB are parallel, then the point Q may be considered as the point at infinity. Then, by the interpretation $AQ = BQ = 0$, the identity is valid as

$$\frac{AP}{0} = \frac{BP}{0} = 0.$$

For the tangential function, note that:

In the formula

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}, \tag{5.1}$$

for $\theta = \pi$, we have: 0=0/0.

In the formula

$$\tan z_1 \pm \tan z_2 = \frac{\sin(z_1 + z_2)}{\sin z_1 \sin z_2}, \tag{5.2}$$

for $z_1 = \pi/2, z_2 = 0$, we have: 0=1/0.

In the elementary identity

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}, \tag{5.3}$$

for the case $\alpha = \beta = \pi/2$, we have

$$\tan\frac{\pi}{2} = \frac{1+1}{1-1\cdot 1} = \frac{2}{0} - 0. \tag{5.4}$$

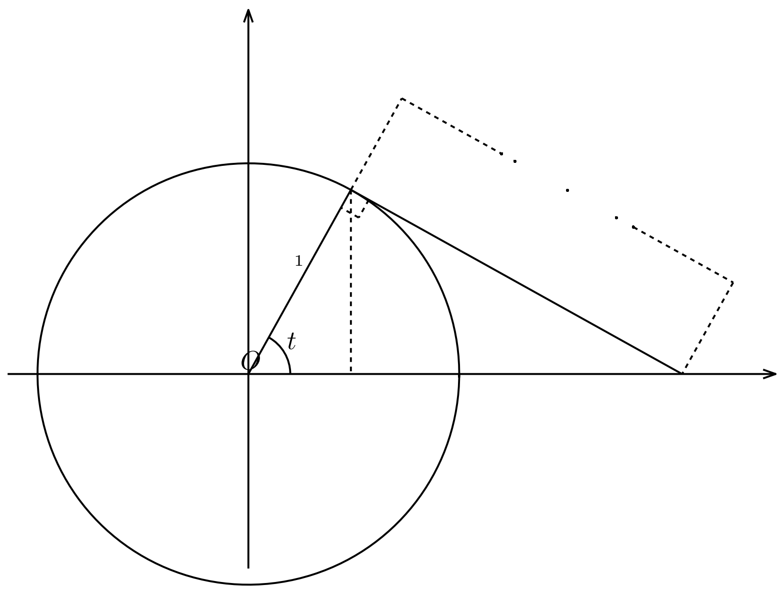


Figure 8.

# 6 Euclidean spaces and division by zero

In this section, we will see the division by zero properties on the Euclidean spaces. Since the impact of the division by zero and division by zero calculus is widely expanded in elementary mathematics, here, elementary topics will be introduced as the first stage.

## 6.1 Broken phenomena of figures by area and volume

The strong discontinuity of the division by zero around the point at infinity will be appeared as the broken of various figures. These phenomena may be looked in many situations as the universe one. However, the simplest cases are disc and sphere (ball) with radius $1/R$. When $R \to +0$, the areas and volumes of discs and balls tend to $+\infty$, respectively, however, when $R = 0$, they are zero, because they become the half-plane and half-space, respectively. These facts may be also looked by analytic geometry, as we see later. However, the results are clear already from the definition of the division by zero:

For this fact, note the following:

The behavior of the space around the point at infinity may be considered by that of the origin by the linear transform $W = 1/z$ (see [2]). We thus see that

$$\lim_{z\to\infty} z = \infty, \tag{6.1}$$

however,

$$[z]_{z=\infty} = 0, \tag{6.2}$$

by the division by zero. Here, $[z]_{z=\infty}$ denotes the value of the function $W = z$ at the topological point at the infinity in one point compactification by Aleksandrov. The difference of (6.1) and (6.2) is very important as we see clearly by the function $W = 1/z$ and the behavior at the origin. The limiting value to the origin and the value at the origin are different. For surprising results, we will state the property in the real space as follows:

$$\lim_{x\to+\infty} x = +\infty, \quad \lim_{x\to-\infty} x = -\infty, \tag{6.3}$$

however,

$$[x]_{+\infty} = 0, \quad [x]_{-\infty} = 0. \tag{6.4}$$

Of course, two points $+\infty$ and $-\infty$ are the same point as the point at infinity. However, $\pm$ will be convenient in order to show the approach directions. In [11], we gave many examples for this property.

In particular, in $z \to \infty$ in (6.1), $\infty$ represents the topological point on the Riemann sphere, meanwhile $\infty$ in the left hand side in (6.1) represents the limit by means of the $\epsilon$ - $\delta$ logic.

## 6.2 Parallel lines

We write lines by

$$L_k : a_k x + b_k y + c_k = 0, k = 1, 2. \tag{6.5}$$

The common point is given by, if $a_1 b_2 - a_2 b_1 \neq 0$; that is, the lines are not parallel

$$\left( \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1} \right). \tag{6.6}$$

By the division by zero, we can understand that if $a_1 b_2 - a_2 b_1 = 0$, then the commom point is always given by

$$(0, 0), \tag{6.7}$$

even the two lines are the same. This fact shows that the image of the Euclidean space is right, because any line is extended to the point at infinity and the point is represented by zero; the origin.

In particular, note that the concept of parallel lines is very important in the Euclidean plane and non-Euclidean geometry. In our sense, there is no parallel line and all lines pass the origin. This will be our world in the Euclidean plane. However, this property is not geometrical and has a strong discontinuity. This surprising property may be looked clearly by the polar representation of a line.

We write a line by the polar coordinate

$$r = \frac{d}{\cos(\theta - \alpha)}, \tag{6.8}$$

where $d = \overline{OH} > 0$ is the distance of the origin O and the line such that OH and the line is orthogonal and H is on the line, $\alpha$ is the angle of the line OH and the positive $x$ axis, and $\theta$ is the angle OP ($P = (r, \theta)$ on the line) and the positive $x$ axis. Then, if $\theta - \alpha = \pi/2$: that is, OP and the line is parallel and P is the point at infinity, then we see that $r = 0$ by the division by zero calculus; the point at infinity is represented by zero and we can consider that the line passes the origin, however, it is in a discontinuous way.

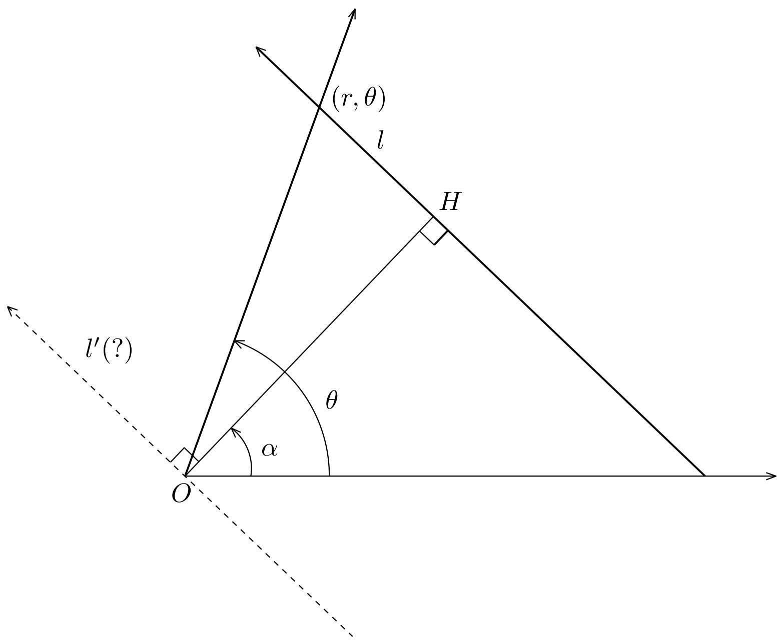


Figure 9.

This will mean simply that any line arrives at the point at infinity and the point is represented by zero and so, for the line we can add the point at the origin. In this sense, we can add the origin to any line as the point of the compactification of the line. This surprising new property may be looked in our mathematics grobally.

The distance $d$ from the origin to the line determined by the two planes

$$\Pi_k : a_k x + b_k y + c_k z = 1, k = 1, 2, \tag{6.9}$$

is given by

$$d = \sqrt{\frac{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2}{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}}. \tag{6.10}$$

If the two planes are coincident, then $d = 0$. Further, if the two planes are parallel, by the division by zero, $d = 0$. This will mean that any plane contains the origin as in a line.

### 6.3 Tangential lines and $\tan \frac{\pi}{2} = 0$

We looked the very fundamental and important formula $\tan \frac{\pi}{2} = 0$ in Section 5. In this subsection, for its importance we will furthermore see its various geometrical meanings.

We consider the high $\tan \theta \left(0 \leq \theta \leq \frac{\pi}{2}\right)$ that is given by the common point of two lines $y = (\tan \theta)x$ and $x = 1$ on the $(x, y)$ plane. Then,

$$\tan \theta \longrightarrow \infty; \quad \theta \longrightarrow \frac{\pi}{2}.$$

However,

$$\tan \frac{\pi}{2} = 0,$$

by the division by zero. The result will show that, when $\theta = \pi/2$, two lines $y = (\tan \theta)x$ and $x = 1$ do not have a common point, because they are parallel in the usual sense. However, in the sense of the division by zero, parallel lines have the common point $(0, 0)$. Therefore, we can see the result $\tan \frac{\pi}{2} = 0$ following our new space idea.

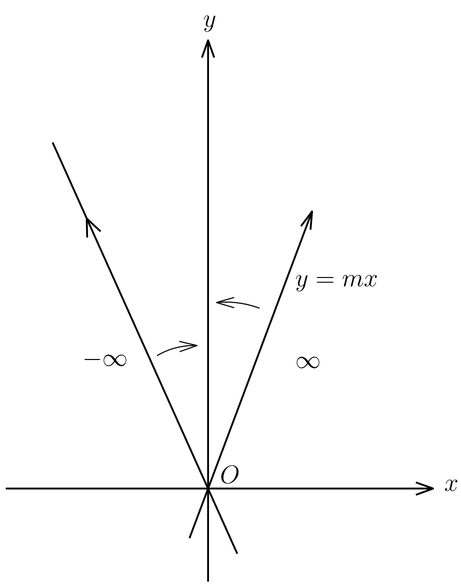


Figure 10.

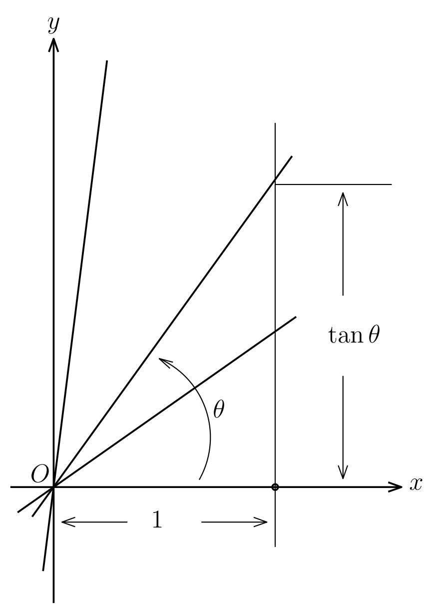


Figure 11.

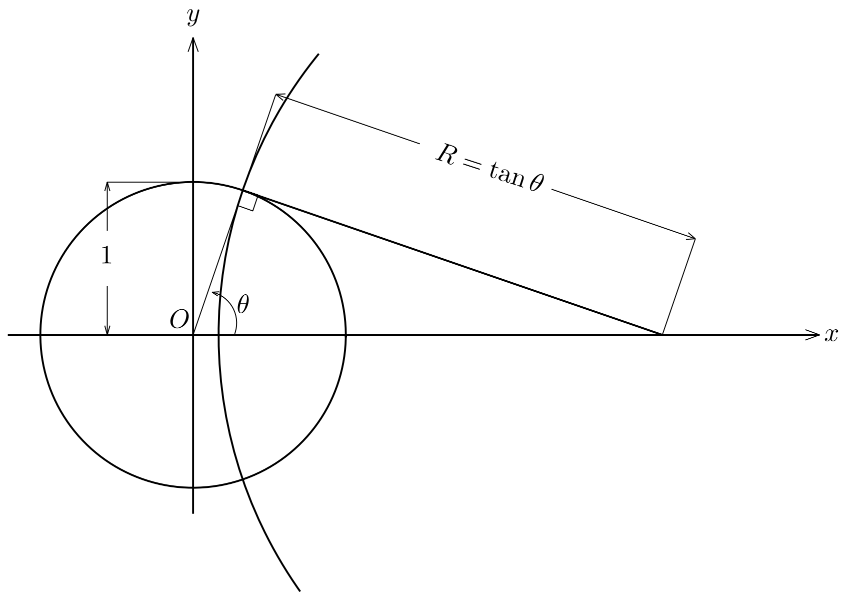


Figure 12.

We consider general lines represented by

$$ax + by + c = 0, a'x + b'y + c' = 0. \tag{6.11}$$

The gradients are given by

$$k = -\frac{a}{b}, k' = -\frac{a'}{b'}, \tag{6.12}$$

respectively. In particular, note that if $b = 0$, then $k = 0$, by the division by zero.

If $kk' = -1$, then the lines are orthogonal; that is,

$$\tan\frac{\pi}{2} = 0 = \pm\frac{k - k'}{1 + kk'}, \tag{6.13}$$

which shows that the division by zero $1/0 = 0$ and orthogonality meets in a very good way.

Furthermore, even in the case of polar coordinates $x = r\cos\theta, y = r\sin\theta$, we can see the division by zero

$$\tan\frac{\pi}{2} = \frac{y}{0} = 0. \tag{6.14}$$

In particular, note that:
From the expansion

$$\tan z = -\sum_{\nu=-\infty}^{+\infty}\left(\frac{1}{z-(2\nu-1)\pi/2}+\frac{1}{(2\nu-1)\pi/2}\right), \tag{6.15}$$

$$\tan\frac{\pi}{2} = 0.$$

The division by zero may be looked even in the rotation of the coordinates.
We will consider a 2 dimensional curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{6.16}$$

and a rotation defined by

$$x = X\cos\theta - Y\sin\theta, \quad y = X\sin\theta + Y\cos\theta. \tag{6.17}$$

Then, we write, by inserting these $(x, y)$

$$AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0. \tag{6.18}$$

Then,

$$H = 0 \Longleftrightarrow \tan 2\theta = \frac{2h}{a-b}. \tag{6.19}$$

If $a = b$, then, by the division by zero,

$$\tan\frac{\pi}{2} = 0, \quad \theta = \frac{\pi}{4}. \tag{6.20}$$

For $h^2 > ab$, the equation

$$ax^2 + 2hxy + by^2 = 0 \tag{6.21}$$

represents 2 lines and the angle $\theta$ made by two lines is given by

$$\tan\theta = \pm\frac{2\sqrt{h^2-ab}}{a+b}. \tag{6.22}$$

If $h^2 - ab = 0$, then, of course, $\theta = 0$. If $a + b = 0$, then, by the division by zero, $\theta = \pi/2$ from $\tan\theta = 0$.

For a hyperbolic function

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad a, b > 0 \tag{6.23}$$

the angle $\theta$ maden by the two asymptotic lines $y = \pm(b/a)x$ is given by

$$\tan\theta = \frac{2(b/a)}{1 - (b/a)^2}. \tag{6.24}$$

If $a = b$, then $\theta = \pi/2$ from $\tan\theta = 0$.

We consider the unit circle with center at the origin on the $(x, y)$ plane. We consider the tangential line for the unit circle at the point that is the common point of the unit circle and the line $y = (\tan\theta)x \left(0 \le \theta \le \frac{\pi}{2}\right)$. Then, the distance $R_\theta$ between the common point and the common point of the tangential line and $x$-axis is given by

$$R_\theta = \tan\theta.$$

Then,

$$R_0 = \tan 0 = 0,$$

and

$$\tan\theta \longrightarrow \infty; \quad \theta \longrightarrow \frac{\pi}{2}.$$

However,

$$R_{\pi/2} = \tan\frac{\pi}{2} = 0.$$

This example shows also that by the stereoprojection mapping of the unit sphere with center the origin $(0, 0, 0)$ onto the plane, the north pole corresponds to the origin $(0, 0)$.

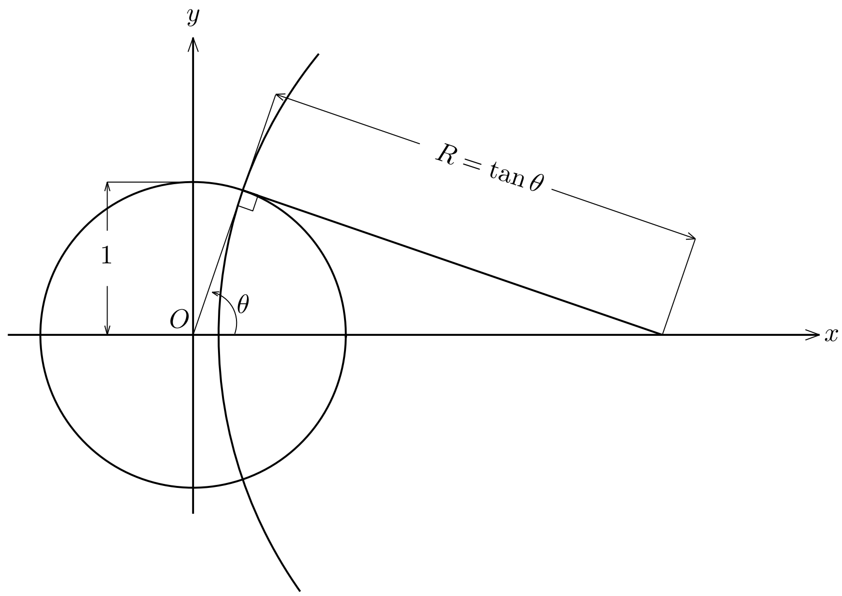


Figure 13.

In this case, we consider the orthogonal circle $C_{R_\theta}$ with the unit circle through at the common point and the symmetric point with respect to the $x$-axis with center $((\cos\theta)^{-1}, 0)$. Then, the circle $C_{R_\theta}$ is as follows:

$C_{R_0}$ is the point $(1, 0)$ with curvature zero, and $C_{R_{\pi/2}}$ (that is, when $R_\theta = \infty$, in the common sense) is the $y$-axis and its curvature is also zero. Meanwhile, by the division by zero, for $\theta = \pi/2$ we have the same result, because $(\cos(\pi/2))^{-1} = 0$.

Note that from the expansion

$$\frac{1}{\cos z} = 1 + \sum_{\nu=-\infty}^{+\infty} (-1)^\nu \left( \frac{1}{z - (2\nu - 1)\pi/2} + \frac{2}{(2\nu - 1)\pi} \right), \tag{6.25}$$

$$\left( \frac{1}{\cos z} \right) \left( \frac{\pi}{2} \right) = 1 - \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{2\nu + 1} = 0.$$

The point $(\cos\theta, 0)$ and $((\cos\theta)^{-1}, 0)$ are the symmetric points with respective to the unit circle, and the origin corresponds to the origin.

In particular, the formal calculation

$$\sqrt{1 + R_{\pi/2}^2} = 1 \tag{6.26}$$

is not good. The identity $\cos^2\theta + \sin^2\theta = 1$ is valid always, however $1 + \tan^2\theta = (\cos\theta)^{-2}$ is not valid for $\theta = \pi/2$.

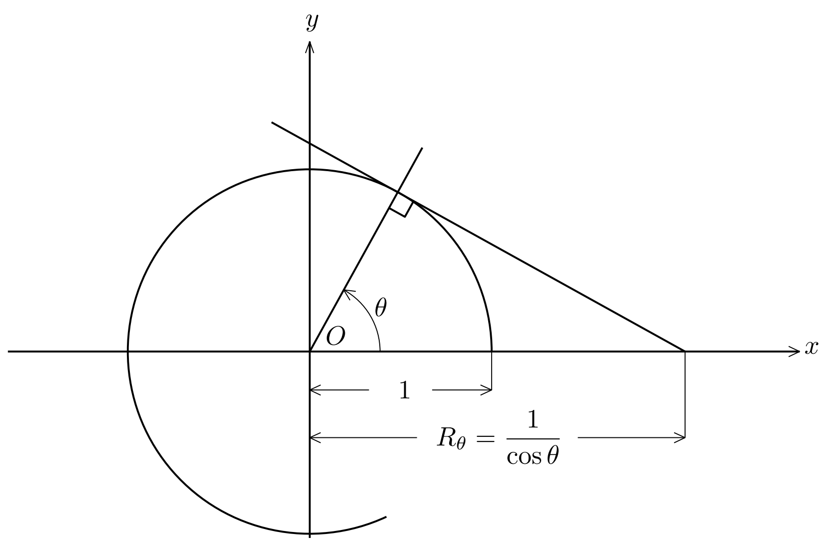


Figure 14.

Note that from the expansin

$$\frac{1}{\cos^2 z} = \sum_{\nu=-\infty}^{+\infty} \frac{1}{(z-(2\nu-1)\pi/2)^2}, \tag{6.27}$$

$$\left(\frac{1}{\cos^2 z}\right)\left(\frac{\pi}{2}\right) = \frac{2}{\pi^2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2} = \frac{1}{3}.$$

On the point $(p,q)(0 \le p, q \le 1)$ on the unit circle, we consider the tangential line $L_{p,q}$ of the unit circle. Then, the common points of the line $L_{p,q}$ with $x$-axis and $y$-axis are $(1/p, 0)$ and $(0, 1/q)$, respectively. Then, the area $S_p$ of the triangle formed by the three points $(0,0)$, $(1/p, 0)$ and $(0, 1/q)$ is given by

$$S_p = \frac{1}{2pq}.$$

Then,

$$p \longrightarrow 0; \quad S_p \longrightarrow +\infty,$$

however,

$$S_0 = 0$$

(H. Michiwaki: 2015.12.5.).

We denote the point on the unit circle on the $(x, y)$ with $(\cos\theta, \sin\theta)$ for the angle $\theta$ with the positive real line. Then, the tangential line of the unit circle at the point meets at the point $(R_\theta, 0)$ for $R_\theta = [\cos\theta]^{-1}$ with the $x$-axis for the case $\theta \neq \pi/2$. Then,

$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty, \tag{6.28}$$

$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty, \tag{6.29}$$

however,

$$R_{\pi/2} = \left[\cos\left(\frac{\pi}{2}\right)\right]^{-1} = 0, \tag{6.30}$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$ (H. Michiwaki: 2015.12.5.).

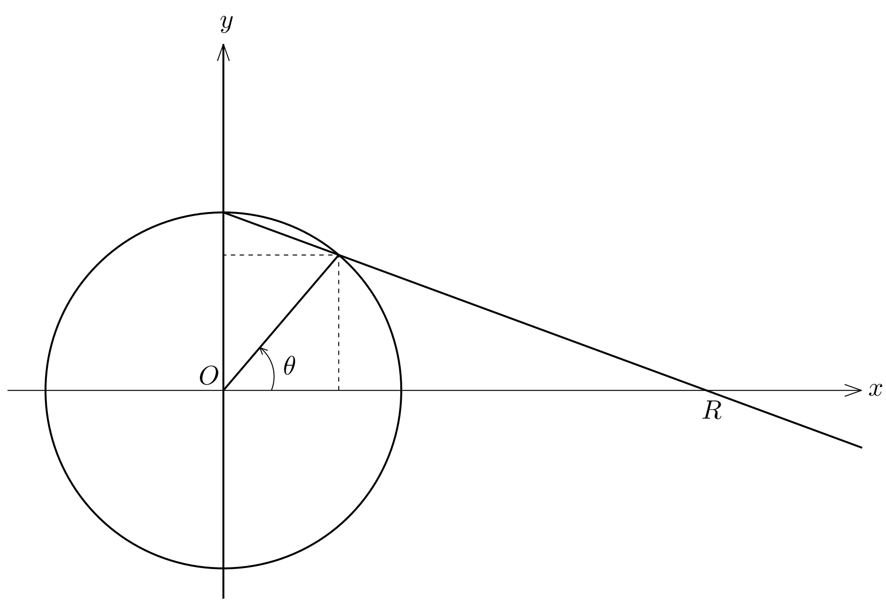


Figure 15.

The line through the points $(0, 1)$ and $(\cos\theta, \sin\theta)$ meets the $x$ axis with the point $(R_\theta, 0)$ for the case $\theta \neq \pi/2$ by

$$R_\theta = \frac{\cos\theta}{1 - \sin\theta}. \tag{6.31}$$

Then,

$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty, \tag{6.32}$$

$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty, \tag{6.33}$$

however,

$$R_{\pi/2} = 0, \tag{6.34}$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$.

Note also that

$$\left[1 - \sin\left(\frac{\pi}{2}\right)\right]^{-1} = 0.$$

For the parabolic equation $y^2 = 4ax, a > 0$, at a point $(x, y)$, the normal line shadow on the $x$-axis is given by

$$|yy'| = 2a. \tag{6.35}$$

At the origin, we have, from $y'(0) = 0$,

$$|yy'| = 0. \tag{6.36}$$

## 6.4 Two Circles

We consider two circles with radii $a, b > 0$ with centers $(a, 0)$ and $(-b, 0)$, respectively. Then, the external common tangent $L_{a,b}$ (we assume that $a < b$) meets the $x$-axis in point $(R_a, 0)$ which is given by, by fixing $b$

$$R_a = \frac{2ab}{b-a}. \tag{6.37}$$

We consider the circle $C_{R_a}$ with center at $(R_a, 0)$ with radius $R_a$ (see Figure 15). Then,

$$a \to b \Longrightarrow R_a \to \infty,$$

however, when $a = b$, then we have $R_b = -2b$ by the division by zero, from the identity

$$\frac{2ab}{b-a} = -2b - \frac{2b^2}{a-b}.$$

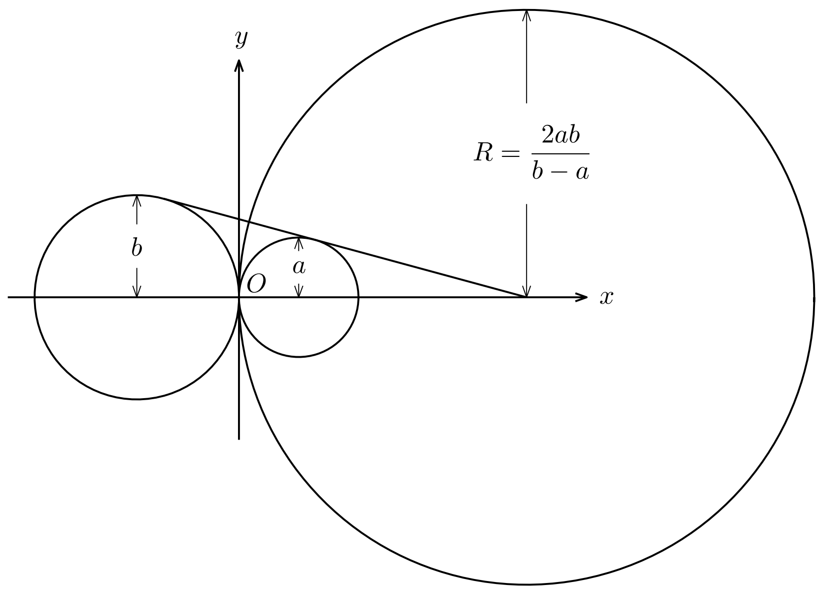


Figure 16.

Meanwhile, when we interpret (6.37) as

$$R_a = \frac{-1}{a-b} \cdot 2ab,$$

we have, for $a = b$, $R_b = 0$. It means that the circle $C_{R_b}$ is the $y$ axis with curvature zero through the origin $(0,0)$.

The above formulas will show strong discontinuity for the change of the $a$ and $b$ from $a = b$ (H. Okumura: 2015.10.29.).

We denote the circles $S_j$:

$$(x - a_j)^2 + (y - b_j)^2 = r_j^2.$$

Then, the common point $(X, Y)$ of the co- and exterior tangential lines of the circles $S_j$ for $j = 1, 2$,

$$(X, Y) = \left( \frac{r_1 a_2 - r_2 a_1}{r_1 - r_2}, \frac{r_1 b_2 - r_2 b_1}{r_1 - r_2} \right).$$

We will fix the circle $S_2$. Then, from the expansion

$$\frac{r_1 a_2 - r_2 a_1}{r_1 - r_2} = \frac{r_2(a_2 - a_1)}{r_1 - r_2} + a_2 \tag{6.38}$$

for $r_1 = r_2$, by the division by zero, we have

$$(X, Y) = (a_2, b_2).$$

Meanwhile, when we interpret (6.38) as

$$\frac{r_1 a_2 - r_2 a_1}{r_1 - r_2} = \frac{1}{r_1 - r_2} \cdot (r_1 a_2 - r_2 a_1),$$

we obtain that

$$(X, Y) = (0, 0),$$

that is reasonable. However, the both cases, the results show strong discontinuity.

## 6.5 Circles and curvature - an interpretation of the division by zero $r/0 = 0$

We consider a solid body called right circular cone whose bottom is a disc with radius $r_2$. We cut the body with a disc of radius $r_1 (0 < r_1 < r_2)$ that is parallel to the bottom disc. We denote the distance by $d$ between the both discs and $R$ the distance between the top point of the cone and the bottom circle on the surface of the cone. Then, $R$ is calculated by Eko Michiwachi (8 years old daughter of Mr. H. Michiwaki ) as follows:

$$R = \frac{r_2}{r_2 - r_1} \sqrt{d^2 + (r_2 - r_1)^2},$$

that is called *EM radius*, because by the rotation of the cone on the plane, the bottom circle writes the circle of radius $R$. We denote by $K = K(R) = 1/R$ the curvature of the circle with radius $R$. We fix the distance $d$. Now note that:

$$r_1 \to r_2 \Longrightarrow R \to \infty.$$

This will be natural in the sense that when $r_1 = r_2$, the circle with radius $R$ becomes a line.

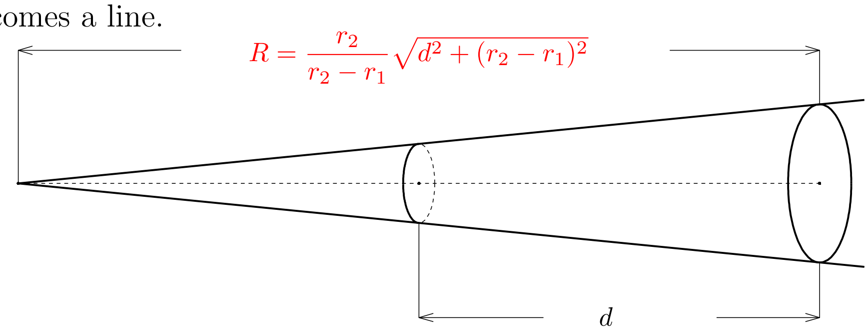

Figure 17.

However, the division by zero will mean that when $r_1 = r_2$, the above EM radius formula makes sense and $R = 0$. What does it mean? Here, note that, however, then the curvature $K = K(0) = 0$ by the division by zero; that is, the circle with radius $R$ becomes a line, similarly. The curvature of a point (circle of radius zero) is zero.

## 6.6 Newton's method

The Newton's method is fundamental when we look for the solutions for some general equation $f(x) = 0$ numerically and practically. We will refer to its prototype case.

We will assume that a function $y = f(x)$ belongs to $C^1$ class. We consider the sequence $\{x_n\}$ for $n = 0, 1, 2, \ldots, n, \ldots$, defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots. \tag{6.39}$$

When $f(x_n) = 0$, we have

$$x_{n+1} = x_n, \tag{6.40}$$

in the reasonable way. Even the case $f'(x_n) = 0$, we have also the reasonable result (6.46), by the division by zero.

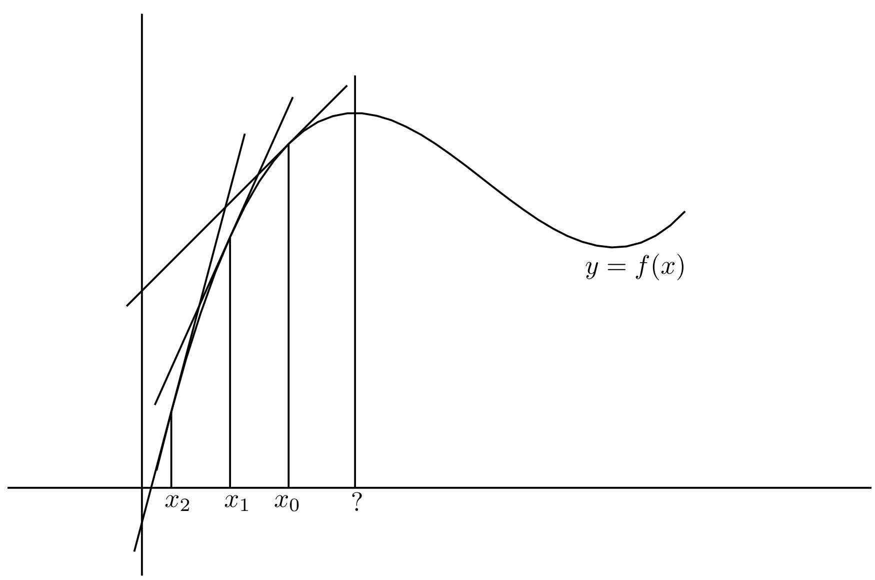

Figure 18.

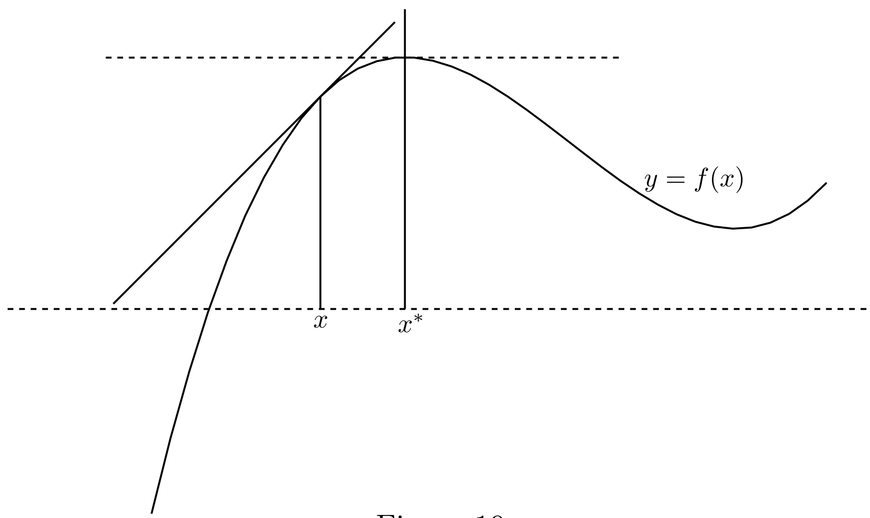


Figure 19.

## 6.7 Cauchy's mean value theorem

For the Cauchy mean value theorem: for $f, g \in Differ(a, b)$, differentiable, and $\in C^0[a, b]$, continuous and if $g(a) \neq g(b)$ and $f'(x)^2 + g'(x)^2 \neq 0$, then there exists $\xi \in (a, b)$ satisfying that

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(\xi)}{g'(\xi)}, \tag{6.41}$$

we do not need the assumptions $g(a) \neq g(b)$ and $f'(x)^2 + g'(x)^2 \neq 0$, by the division by zero. Indeed, if $g(a) = g(b)$, then, by the Rolle theorem, there exists $\xi \in (a, b)$ such that $g'(\xi) = 0$. Then, the both terms are zero and the equality is valid.

For $f, g \in C^2[a, b]$, there exists a $\xi \in (a, b)$ satisfying

$$\frac{f(b) - f(a) - (b - a)f'(a)}{g(b) - g(a) - (b - a)g'(a)} = \frac{f''(a)}{g''(a)}.$$

Here, we do not need the assumption

$$g(b) - g(a) - (b - a)g'(a) \neq 0,$$

by the division by zero.

## 6.8 Length of tangential lines

We will consider a function $y = f(x)$ of $C^1$ class on the real line. We consider the tangential line through $(x, f(x))$

$$Y = f'(x)(X - x) + f(x). \tag{6.42}$$

Then, the length (or distance) $d(x)$ between the point $(x, f(x))$ and $\left(x - \frac{f(x)}{f'(x)}, 0\right)$ is given by, for $f'(x) \neq 0$

$$d(x) = |f(x)|\sqrt{1 + \frac{1}{f'(x)^2}}. \tag{6.43}$$

How will be the case $f'(x^*) = 0$? Then, the division by zero shows that

$$d(x^*) = |f(x^*)|. \tag{6.44}$$

Meanwhile, the $x$ axis point $(X_t, 0)$ of the tangential line at $(x, y)$ and $y$ axis point $(0, Y_n)$ of the normal line at $(x, y)$ are given by

$$X_t = x - \frac{f(x)}{f'(x)} \tag{6.45}$$

and

$$Y_n = y + \frac{x}{f'(x)}, \tag{6.46}$$

respectively. Then, if $f'(x) = 0$, we obtain the reasonable results:

$$X_t = x, \quad Y_n = y. \tag{6.47}$$

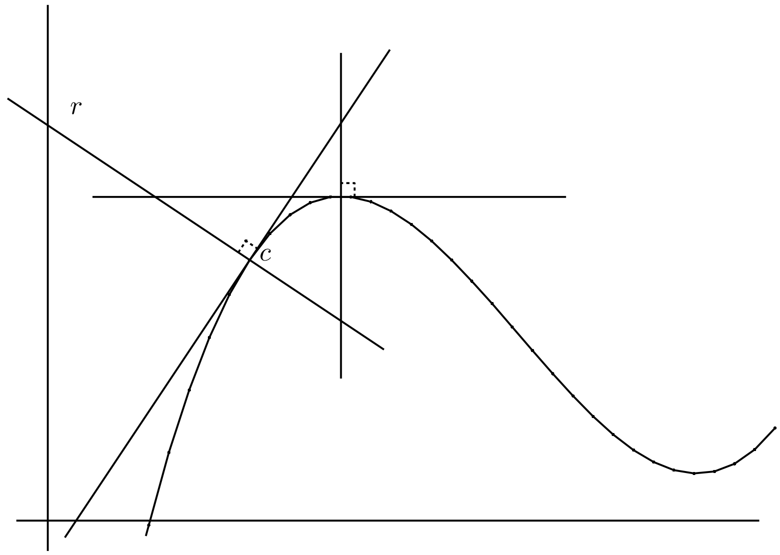


Figure 20.

## 6.9 Curvature and center of curvature

We will assume that a function $y = f(x)$ is of class $C^2$. Then, the curvature radius $\rho$ and the center $O(x, y)$ of the curvature at point $(x, f(x))$ are given by

$$\rho(x, y) = \frac{(1 + (y')^2)^{3/2}}{y''} \tag{6.48}$$

and

$$O(x, y) = \left( x - \frac{1 + (y')^2}{y''} y', y + \frac{1 + (y')^2}{y''} \right), \tag{6.49}$$

respectively. Then, if $y'' = 0$, we have:

$$\rho(x, y) = 0 \tag{6.50}$$

and

$$O(x, y) = (x, y), \tag{6.51}$$

by the division by zero. They are reasonable.

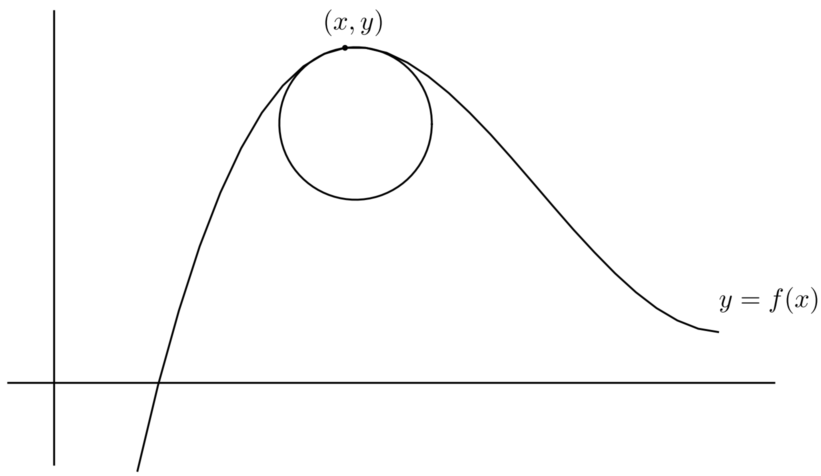


Figure 21.

We will consider a curve $\mathbf{r} = \mathbf{r}(s), s = s(t)$ of class $C^2$. Then,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \mathbf{t} = \frac{d\mathbf{r}(\mathbf{s})}{ds}, v = \frac{ds}{dt}, \frac{d\mathbf{t}(\mathbf{s})}{ds} = \frac{1}{\rho}\mathbf{n},$$

by the principal normal unit vector $\mathbf{n}$. Then, we see that

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{t} + \frac{v^2}{\rho}\mathbf{n}.$$

If $\rho(s_0) = 0$, then

$$\mathbf{a}(s_0) = \left[\frac{dv}{dt}\mathbf{t}\right]_{s=s_0} \tag{6.52}$$

and

$$\left[\frac{v^2}{\rho}\right]_{s=s_0} = \infty \tag{6.53}$$

will be funny. It will be the zero.

## 6.10 $n = 2, 1, 0$ regular polygons inscribed in a disc

We consider $n$ regular polygons inscribed in a fixed disc with radius $a$. Then we note that their area $S_n$ and the lengths $L_n$ of the sum of the sides are given by

$$S_n = \frac{na^2}{2} \sin \frac{2\pi}{n} \tag{6.54}$$

and

$$L_n = 2na \sin \frac{\pi}{n}, \tag{6.55}$$

respectively (see Figure 21). For $n \geq 3$, the results are clear.

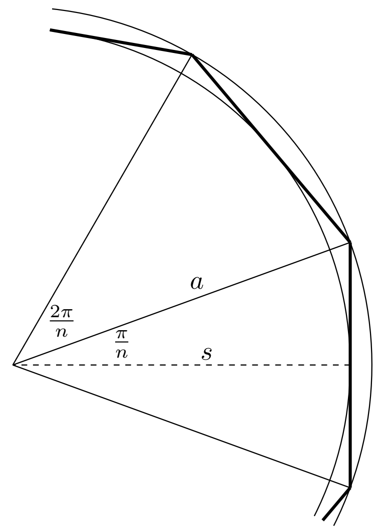


Figure 22.

For $n = 2$, we will consider two diameters that are the same. We can consider it as a generalized regular polygon inscribed in the disc as a degenerate case. Then, $S_2 = 0$ and $L_2 = 4a$, and the general formulas are valid.

Next, we will consider the case $n = 1$. Then the corresponding regular polygon is a just diameter of the disc. Then, $S_1 = 0$ and $L_1 = 0$ that will mean that any regular polygon inscribed in the disc may not be formed and so its area and length of the side are zero.

For a $n = 1$ triangle, if 1 means one side, then we can interpretate as in the above, however, if we consider 1 as one vertex, the above situation may be consider as one point on the circle which coincides with $S_l = L_l = 0$.

Now we will consider the case $n = 0$. Then, by the division by zero calculus, we obtain that $S_0 = \pi a^2$ and $L_0 = 2\pi a$. Note that they are the area and the length of the disc. How to understand the results? Imagine contrary $n$ tending to infinity, then the corresponding regular polygons inscribed in the disc tend to the disc. Recall our new idea that the point at infinity is represented by 0. Therefore, the results say that $n = 0$ regular polygons are $n = \infty$ regular polygons inscribed in the disc in a sense and they are the disc. This is our interpretation of the theorem:

**Theorem.** $n = 0$ *regular polygons inscribed in a disc are the whole disc.*

In addition, note that each inner angle $A_n$ of a general $n$ regular polygon inscribed in a fixed disc with radius $a$ is given by

$$A_n = \left(1 - \frac{2}{n}\right)\pi. \tag{6.56}$$

The circumstances are similar for $n$ regular polygons circumscribed in the disc, because the corresponding data are given by

$$S_n = na^2 \tan\frac{\pi}{n} \tag{6.57}$$

and

$$L_n = 2na \tan\frac{\pi}{n}, \tag{6.58}$$

and (6.63), respectively.

## 6.11 Our life figure

As an interesting figure which shows an interesting relation between 0 and infinity, we will consider a sector $\Delta_\alpha$ on the complex $z = x + iy$ plane

$$\Delta_\alpha = \left\{|\arg z| < \alpha; 0 < \alpha < \frac{\pi}{2}\right\}.$$

We will consider a disc inscribed in the sector $\Delta_\alpha$ whose center $(k, 0)$ with radius $r$. Then, we have

$$r = k \sin\alpha. \tag{6.59}$$

Then, note that as $k$ tends to zero, $r$ tends to zero, meanwhile $k$ tends to $+\infty$, $r$ tends to $+\infty$. However, by our division by zero calculus, we see that immediately that

$$[r]_{r=\infty} = 0. \tag{6.60}$$

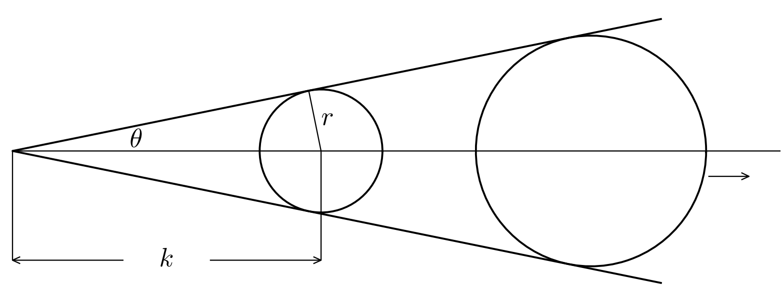


Figure 23: $\theta$: const, $r \to \infty$

On the sector, we see that from the origin as the point 0, the inscribed discs are increasing endlessly, however their final disc reduces to the origin suddenly - it seems that the whole process looks like our life in the viewpoint of our initial and final.

## 6.12 H. Okumura's example

The suprising example by H. Okumura will show a new phenomenon at the point at infinity.

On the sector $\Delta_\alpha$, we shall change the angle and we consider a fixed circle $C_a, a > 0$ with radius $a$ inscribed in the sectors. We see that when the circle tends to $+\infty$, the angles $\alpha$ tend to zero. How will be the case $\alpha = 0$? Then, we will not be able to see the position of the circle. Surprisingly enough, then $C_a$ is the circle with center at the origin 0. This result is derived from the division by zero calculus for the formula

$$k = \frac{a}{\sin \alpha}. \tag{6.61}$$

The two lines $\arg z = \alpha$ and $\arg z = -\alpha$ were tangential lines of the circle $C_a$ and now they are the positive real line. The gradient of the positive real line is of course zero. Note here that the gradient of the positive imaginary line is zero by the division by zero calculus that means $\tan \frac{\pi}{2} = 0$. Therefore, we can understand that the positive real line is still a tangential line of the circle $C_a$.

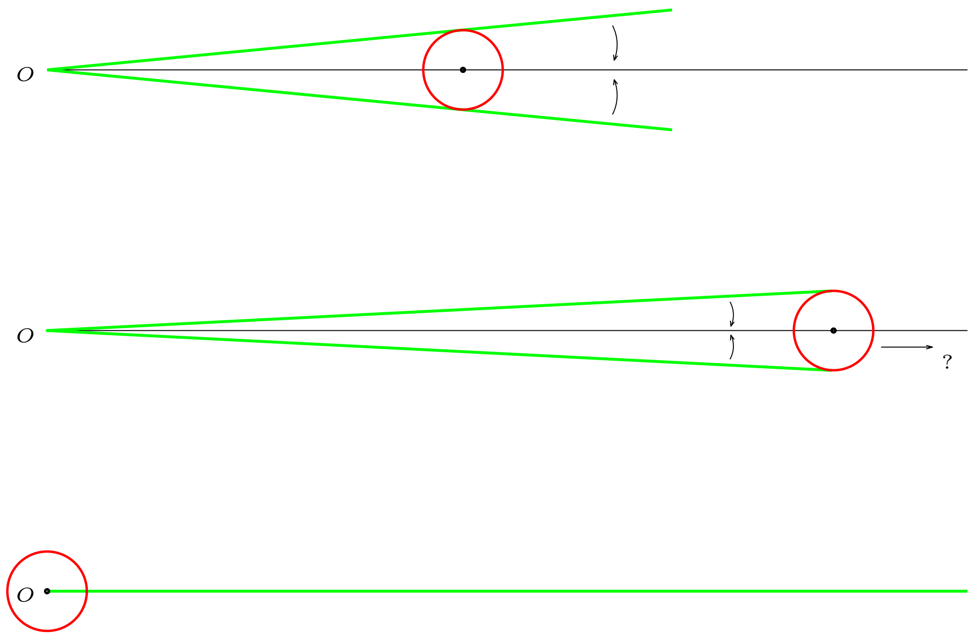


Figure 24.

This will show some great relation between zero and infinity. We can see some mysterious property around the point at infinity.

## 6.13 Interpretation by analytic geometry

For a function
$$S(x,y) = a(x^2+y^2) + 2gx + 2fy + c, \tag{6.62}$$
the radius $R$ of the circle $S(x,y) = 0$ is given by
$$R = \sqrt{\frac{g^2+f^2-ac}{a^2}}. \tag{6.63}$$
If $a = 0$, then the area $\pi R^2$ of the disc is zero, by the division by zero; that is, the circle is a line (degenerate).

The center of the circle (6.68) is given by
$$\left(-\frac{g}{a}, -\frac{f}{a}\right). \tag{6.64}$$
Therefore, the center of a general line
$$2gx + 2fy + c = 0 \tag{6.65}$$

may be considered as the origin $(0,0)$, by the division by zero.

We consider the functions

$$S_j(x,y) = a_j(x^2+y^2) + 2g_j x + 2f_j y + c_j. \tag{6.66}$$

The distance $d$ of the centers of the circles $S_1(x,y)=0$ and $S_2(x,y)=0$ is given by

$$d^2 = \frac{g_1^2+f_1^2}{a_1^2} - 2\frac{g_1g_2+f_1f_2}{a_1a_2} + \frac{g_2^2+f_2^2}{a_2^2}. \tag{6.67}$$

If $a_1 = 0$, then by the division by zero

$$d^2 = \frac{g_2^2+f_2^2}{a_2^2}. \tag{6.68}$$

Then, $S_1(x,y)=0$ is a line and its center is the origin $(0,0)$. Therefore, the result is very reasonable.

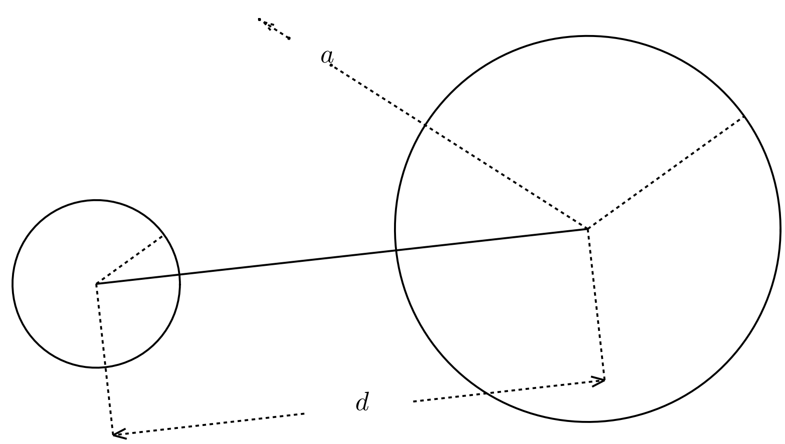


Figure 25.

Meanwhile, the identity $\cos^2\theta + \sin^2\theta = 1$ is valid always, however $1 + \tan^2\theta = (\cos\theta)^{-2}$ is not valid for $\theta = \pi/2$, in the sense of the division by zero, because we consider the formula at $\theta = \pi/2$, with not the limiting values.

# 7 Mirror image with respect to a circle

For simplicity, we will consider the unit circle $|z| = 1$ on the complex $z = x + iy$ plane. Then, we have the reflection formula

$$z^* = \frac{1}{\overline{z}} \tag{7.1}$$

for any point $z$, as well-known ([2]). For the reflection point $z^*$, there is no problem for the points $z \neq 0, \infty$. As the classical result, the reflection of zero is the point at infinity and conversely, for the point at infinity we have the zero point. The reflection is a one to one and onto mapping between the inside and the outside of the unit circle, by considering the point at infinity.

Are these correspondences, however, suitable? Does there exist the point at $\infty$, really? Is the point at infinity corresponding to the zero point, by the reflection? Is the point at $\infty$ reasonable from the practical point of view? Indeed, where can we find the point at infinity? Of course, we know plesantly the point at infinity on the Riemann sphere, however, on the complex $z$-plane it seems that we can not find the corresponding point. When we approach to the origin on a radial line, it seems that the correspondence reflection points approach to *the point at infinity* with the direction (on the radial line).

On the concept of the division by zero, there is no the point at infinity $\infty$ as the numbers. For any point $z$ such that $|z| > 1$, there exists the unique point $z^*$ by (9.1). Meanwhile, for any point $z$ such that $|z| < 1$ except $z = 0$, there exits the unique point $z^*$ by (9.1). Here, note that for $z = 0$, by the division by zero, $z^* = 0$. Furthermore, we can see that

$$\lim_{z \to 0} z^* = \infty, \tag{7.2}$$

however, for $z = 0$ itself, by the division by zero, we have $z^* = 0$. This will mean a strong discontinuity of the functions $W = \frac{1}{z}$ and (9.1) at the origin $z = 0$; that is a typical property of the division by zero. This strong discontinuity may be looked in the above reflection property, physically.

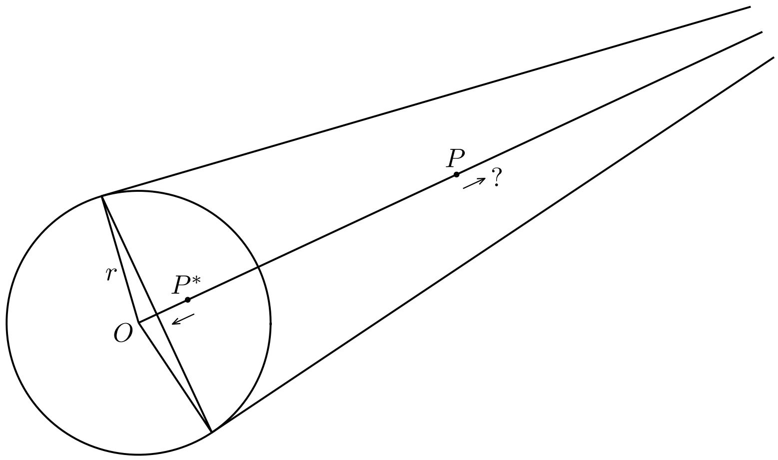


Figure 26.

The result is a surprising one in a sense; indeed, by considering the geometrical corresponding of the mirror image, we will consider the center corresponds to the point at infinity that is represented by the origin $z = 0$. This will show that the mirror image is not followed by this concept; the corresponding seems to come from the concept of one-to-one and onto mapping.

**Should we exclude the point at infinity, from the numbers?** We were able to look the strong discontinuity of the division by zero in the reflection with respect to circles, physically ( geometrical optics ). The division by zero gives a one to one and onto mapping of the reflection (9.1) from the whole complex plane onto the whole complex plane.

**The infinity $\infty$ may be considered as in (9.2) as the usual sense of limits,** however, the infinity $\infty$ is not a definite number.

On the $x, y$ plane, we shall consider the inversion relation with respect to the circle with radius $R$ and with center at the origin:

$$x' = \frac{xR^2}{x^2 + y^2}, \quad y' = \frac{yR^2}{x^2 + y^2}. \tag{7.3}$$

Then, the line

$$ax + by + c = 0 \tag{7.4}$$

is transformed to the line

$$R^2(ax' + by') + c((x')^2 + (y')^2) = 0. \tag{7.5}$$

In particular, for $c = 0$, the line $ax + by = 0$ is transformed to the line $ax' + by' = 0$. This corresponding is one-to-one and onto, and so the origin $(0, 0)$ have to correspond to the origin $(0, 0)$.

For the elliptic curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0 \tag{7.6}$$

and for the similar correspondences

$$x' = \frac{a^2 b^2 x}{b^2 x^2 + a^2 y^2}, \quad y' = \frac{a^2 b^2 y^2}{b^2 x^2 + a^2 y^2}. \tag{7.7}$$

the origin corresponds to itself.

The pole $(x_1, y_1)$ of the line

$$ax + by + c = 0 \tag{7.8}$$

with respect to a circle with radius $R$ with center $(x_0, y_0)$ is given by

$$x_1 = x_0 - \frac{aR^2}{ax_0 + by_0 + c} \tag{7.9}$$

and

$$y_1 = y_0 - \frac{bR^2}{ax_0 + by_0 + c}. \tag{7.10}$$

If $ax_0 + by_0 + c = 0$, then we have $(x_1, y_1) = (x_0, y_0)$.

Furthermore, for various higher dimensional cases the results are similar.

# 8 Stereographic projection

For a great meaning and importance, we will see that the point at infinity is represented by zero.

## 8.1 The point at infinity is represented by zero

By considering the stereographic projection, we will be able to see that the point at infinity is represented by zero.

Consider the sphere $(\xi, \eta, \zeta)$ with radius $1/2$ put on the complex $z = x+iy$ plane with center $(0, 0, 1/2)$. From the north pole $N(0, 0, 1)$, we consider the stereographic projection of the point $P(\xi, \eta, \zeta)$ on the sphere onto the complex $z(= x + iy)$ plane; that is,

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}. \tag{8.1}$$

If $\zeta = 1$, then, by the division by zero, the north pole corresponds to the origin $(0, 0) = 0$.

Here, note that

$$x^2 + y^2 = \frac{\zeta}{1-\zeta}.$$

For $\zeta = 1$, we should consider as $1/0 = 0$, not by the division by zero calculus,

$$\frac{\zeta}{1-\zeta} = -1 - \frac{1}{\zeta - 1}.$$

We will consider the unit sphere $\{(x_1, x_2, x_3); x_1^2+x_2^2+x_3^2 = 1\}$. From the north pole $N(0, 0, 1)$, we consider the stereographic projection of the point $P(x_1, x_2, x_3)$ on the sphere onto the $(x, y)$ plane; that is,

$$(x_1, x_2, x_3) = \tag{8.2}$$

$$\left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{1-1/(x^2+y^2)}{1+1/(x^2+y^2)}\right).$$

Then, we see that the north pole corresponds to the origin.

Next, we will consider the semi-sphere $(\xi, \eta, \zeta)$ with center $C(0, 0, 1)$ on the origin on the $(x, y)$ plane. From the center $C(0, 0, 1)$, we consider the stereographic projection of the point $P(\xi, \eta, \zeta)$ on the semi- sphere onto the complex $(x, y)$ plane; that is,

$$x = \frac{\xi}{1-\zeta}, y = \frac{\eta}{1-\zeta}. \tag{8.3}$$

If $\zeta = 1$, then, by the division by zero, the center $C$ corresponds to the origin $(0, 0)$.

Meanwhile, we will consider the mapping from the open unit disc onto $\mathbf{R}^2$ in one to one and onto

$$\xi = \frac{x\sqrt{x^2+y^2}}{1+x^2+y^2}, \quad \eta = \frac{y\sqrt{x^2+y^2}}{1+x^2+y^2}$$

or

$$x = \frac{\xi}{\sqrt{\rho(1-\rho)}}, y = \frac{\eta}{\sqrt{\rho(1-\rho)}}; \quad \rho^2 = \xi^2 + \eta^2.$$

Note that the point $(x, y) = (0, 0)$ corresponds to $\rho = 0; (\xi, \eta) = (0, 0)$ and $\rho = 1$.

## 8.2 A contradiction of classical idea for $1/0 = \infty$

The infinity $\infty$ may be considered by the idea of the limiting, however, we had considered it as a number, for sometimes, typically, the point at infinity was represented by $\infty$ for some long years. For this fact, we will show a formal contradiction.

We will consider the stereographic projection by means of the unit sphere

$$\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = 1$$

from the complex $z = x + iy$ plane onto the sphere. Then, we obtain the correspondences

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}$$

and

$$\xi = \frac{1}{2}\frac{z+\overline{z}}{z\overline{z}+1}, \eta = \frac{1}{2i}\frac{z-\overline{z}}{z\overline{z}+1}, \zeta = \frac{z\overline{z}}{z\overline{z}+1}.$$

In general, two points $P$ and $Q_1$ on the diameter of the unit sphere correspond to $z$ and $z_1$, respectively if and only if

$$z\overline{z_1} + 1 = 0. \tag{8.4}$$

Meanwhile, two points $P$ and $Q_2$ on the symmetric points on the unit sphere with respect to the plane $\zeta = \frac{1}{2}$ correspond to $z$ and $z_2$, respectively if and only if

$$z\overline{z_2} - 1 = 0. \tag{8.5}$$

If the point $P$ is the origin or the north pole, then the points $Q_1$ and $Q_2$ are the same point. Then, the identities (10.4) and (10.5) are not valid that show a contradiction.

Meanwhile, if we write (10.4) and (10.5)

$$z = -\frac{1}{\overline{z_1}} \tag{8.6}$$

and

$$z = \frac{1}{\overline{z_2}}, \tag{8.7}$$

respectively, we see that the division by zero (1.2) is valid.

## 8.3 Natural meanings of $1/0 = 0$

For constants $a$ and $b$ satisfying

$$\frac{1}{a} + \frac{1}{b} = k, \quad (\neq 0, const.)$$

the function

$$\frac{x}{a} + \frac{y}{b} = 1$$

passes the point $(1/k, 1/k)$. If $a = 0$, then, by the division by zero, $b = 1/k$ and $y = 1/k$; this result is natural.

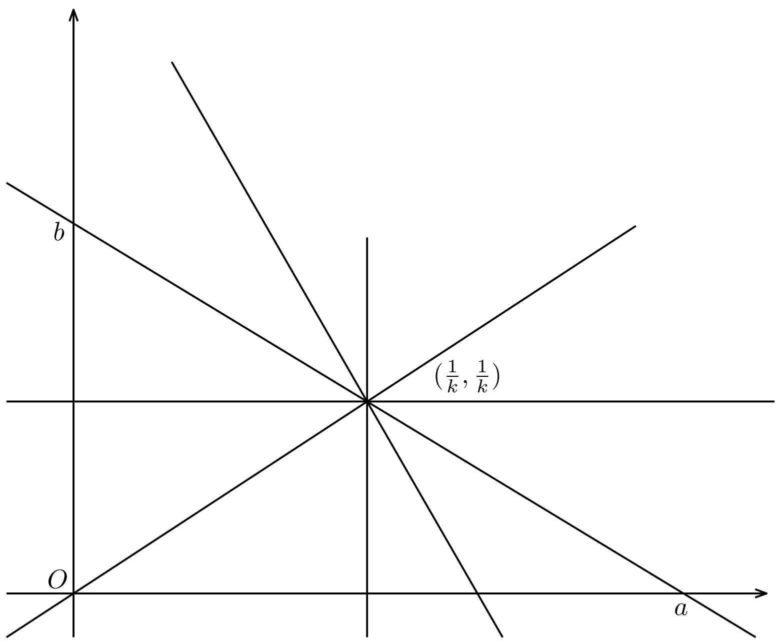

Figure 27.

We will consider the line $y = m(x-a)+b$ through a fixed point $(a, b)$; $a, b > 0$ with gradient $m$. We set $A(0, -am+b)$ and $B(a-(b/m), 0)$ that are common points with the line and both lines $x = 0$ and $y = 0$, respectively. Then,

$$\overline{AB}^2 = (-am + b)^2 + \left(a - \frac{b}{m}\right)^2.$$

If $m = 0$, then $A(0, b)$ and $B(a, 0)$, by the division by zero, and furthermore

$$\overline{AB}^2 = a^2 + b^2.$$

Then, the line AB is a corresponding to the line between the origin and the point $(a, b)$. Note that this line has only one common point with the both lines $x = 0$ and $y = 0$. Therefore, this result will be very natural in a sense. – Indeed, we can understand that the line $\overline{AB}$ is broken as the two lines $(0, b) - (a, b)$ and $(a, b) - (a, 0)$, suddenly.

The general line equation with gradient $m$ is given by, with a constant $b$

$$y = m(x - a) + b \tag{8.8}$$

or

$$\frac{y}{m} = x - a + \frac{b}{m}. \tag{8.9}$$

By $m = 0$, we obtain the equation $x = a$, by the division by zero. This equation may be considered the cases $m = \infty$ and $m = -\infty$, and these cases may be considered by the strictly right logic with the division by zero.

By the division by zero, we can consider the equation (10.8) as a general line equation.

In the Lami's formula for three vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ satisfying

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}, \tag{8.10}$$

$$\frac{\|\mathbf{A}\|}{\sin\alpha} = \frac{\|\mathbf{B}\|}{\sin\beta} = \frac{\|\mathbf{C}\|}{\sin\gamma}, \tag{8.11}$$

if $\alpha = 0$, then we obtain:

$$\frac{\|\mathbf{A}\|}{0} = \frac{\|\mathbf{B}\|}{0} = \frac{\|\mathbf{C}\|}{0} = 0, \tag{8.12}$$

Here, of course, $\alpha$ is the angle of $\mathbf{B}$ and $\mathbf{C}$, $\beta$ is the angle of $\mathbf{C}$ and $\mathbf{A}$, and $\gamma$ is the angle of $\mathbf{A}$ and $\mathbf{B}$,

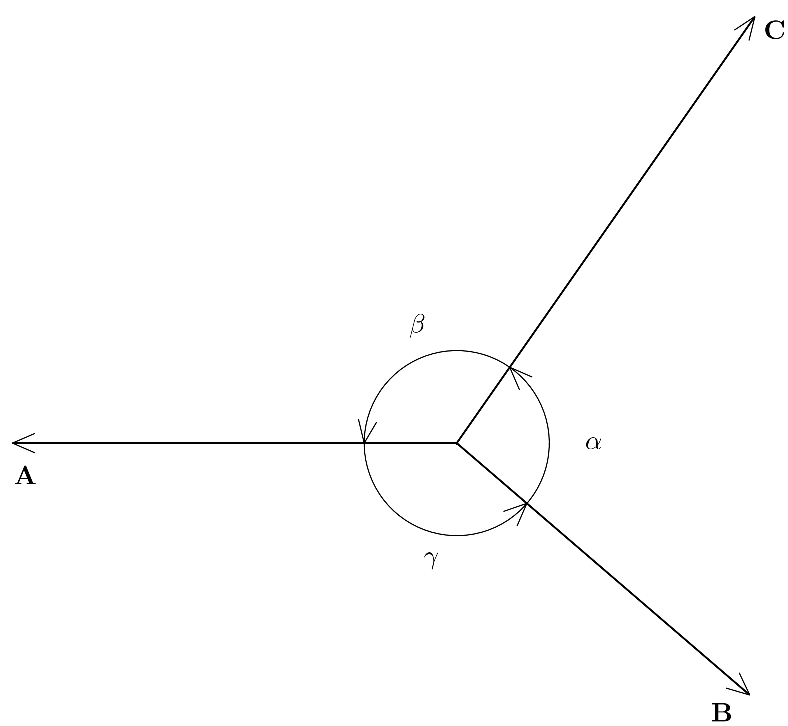

Figure 28.

For the Newton's formula; that is, for a $C^2$ class function $y = f(x)$, the curvature $K$ at the origin is given by

$$K = \lim_{x\to 0} \left| \frac{x^2}{2y} \right| = \left| \frac{1}{f''(0)} \right|, \tag{8.13}$$

we have: for $f''(0) = 0$,

$$K = \frac{1}{0} = 0. \tag{8.14}$$

## 8.4 Double natures of the zero point $z = 0$

Any line on the complex plane arrives at the point at infinity and the point at infinity is represented by zero. That is, a line is, indeed, contains the origin; the true line should be considered as the sum of a usual line and the origin. We can say that it is a compactification of the line and the compacted point is the point at infinity, however, it it is represented by $z = 0$. Later, we will see this property by analytic geometry and the division by zero calculus in many situations.

However, for the general line equation

$$ax + by + c = 0, \tag{8.15}$$

by using the polar coordinates $x = r\cos\theta, y = r\sin\theta$, we have

$$r = \frac{-c}{a\cos\theta + b\sin\theta}. \tag{8.16}$$

When $a\cos\theta + b\sin\theta = 0$, by the division by zero, we have $r = 0$; that is, we can consider that the line contains the origin.

The envelop of the linear lines represented by, for constants $m$ and a fixed constant $p > 0$,

$$y = mx + \frac{p}{m}, \tag{8.17}$$

we have the function, by using an elementary ordinary differential equation,

$$y^2 = 4px. \tag{8.18}$$

The origin of this parabolic function is missing from the envelop of the linear functions, because the linear equations do not contain the $y$ axis as the tangential line of the parabolic function. Now recall that, by the division by zero, as the linear equation for $m = 0$, we have the function $y = 0$, the $x$ axis. Note that both the $x$ axis $y = 0$ and the parabolic function have the zero gradient at the origin; that will mean that in the reasonable sense the $x$ axis is a tangential line of the parabolic function. Anyhow, by the division by zero, the envelop of the linear functions may be considered as the whole parabolic function containing the origin.

When we consider the limiting of the linear equations as $m \to 0$, we will think that the limit function is a parallel line to the $x$ axis through the point at infinity. Since the point at infinity is represented by zero, it will become the $x$ axis.

Meanwhile, when we consider the limiting function as $m \to \infty$, we have the $y$ axis $x = 0$ and this function is an ordinally tangential line of the parabolic function. From these two tangential lines, we see that the origin has **double natures**; one is the continuous tangential line $x = 0$ and the second is the discontinuous tangential line $y = 0$.

In addition, note that the tangential point of (10.18) for the line (10.17) is given by

$$\left( \frac{p}{m}, \frac{2p}{m} \right) \tag{8.19}$$

and it is $(0, 0)$ for $m = 0$.

We can see the point at infinity is reflected to the origin; and so, the origin has the double natures; one is the native origin and another is reflected to the origin of the point at infinity.

# 9 Interesting examples in the division by zero

We will give interesting examples in the division by zero. Indeed, the division by zero may be looked in the elementary mathematics and also in the universe.

- For the line
$$\frac{x}{a} + \frac{y}{b} = 1, \tag{9.1}$$
if $a = 0$, then by the division by zero, we have the line $y = b$. This is a very interesting property creating new phenomena at the term $x/a$ for $a = 0$.

  Note that here we can not consider the case $a = b = 0$.

- For the area $S(a, b) = ab$ of the rectangle with sides of lengths $a, b$, we have
$$a = \frac{S(a, b)}{b} \tag{9.2}$$
and for $b = 0$, formally
$$a = \frac{0}{0}. \tag{9.3}$$
However, there exists a contradiction. $S(a, b)$ depends on $b$ and by the division by zero calculus, we have, for the case $b = 0$, the right result
$$\frac{S(a, b)}{b} = a. \tag{9.4}$$

- For the identity
$$(a^2 + b^2)(a^2 - b^2) = c^2(a^2 - b^2); a, b, c > 0 \tag{9.5}$$
if $a \neq b$, then we have the Pythagorean theorem
$$a^2 + b^2 = c^2. \tag{9.6}$$
However, for the case $a = b$, we have also the Pythagorean theorem, by the division by zero calculus
$$2a^2 = c^2. \tag{9.7}$$

- We consider 4 lines

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_1x + b_1y + c_1' &= 0, \\ a_2x + b_2y + c_2 &= 0, \\ a_2x + b_2y + c_2' &= 0, \end{aligned} \tag{9.8}$$

  Then, the area $S$ surrounded by these lines is given by the formula

$$S = \frac{|c_1 - c_1'| \cdot |c_1 - c_1'|}{|a_1b_2 - a_2b_1|}. \tag{9.9}$$

  Of course, if $|a_1b_2 - a_2b_1| = 0$, then $S = 0$.

- $\frac{1}{\sin 0} = \frac{1}{\cos \pi/2} = 0$. Consider the linear equation with a fixed positive constant $a$

$$\frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1. \tag{9.10}$$

  Then, the results are clear from the graphic meanings.

- For the tangential line at a point $(a\cos\theta, \sin\theta)$ on the elliptic curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0 \tag{9.11}$$

  we have $Q(a/(\cos\theta), 0)$ and $R(0, b/(\sin\theta))$ as the common points with $x$ and $y$ axisis, respectively. if $\theta = 0$, then $Q(a, 0)$ and $R(0, 0)$. If $\theta = \pi/2$, then $Q(0, 0)$ and $R(0, b)$.

- For the tangential line at the point $(a\cos\theta, \sin\theta)$ on the elliptic curve, we shall consider the area $S(\theta)$ of the triangle formed by this line and $x, y$ axises

$$S(\theta) = \frac{ab}{|\sin\theta|}.$$

  Then, by the division by zero calculus, we have $S(0) = 0$.

- The common point of $B$ (resp. $B'$) of a tangential line (7.10) and the line $x = a$ (resp. $x = -a$) is given by

$$B\left(a, \frac{b(1 - \cos\theta)}{\sin\theta}\right).$$

(resp.

$$B'\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right).$$

) The circle with diameter $BB'$ is given by

$$x^2 + y^2 - \frac{2b}{\sin\theta}y - (a^2 - b^2) = 0.$$

Note that this circle passes two forcus points of the elliptic curve. Note that for $\theta = 0$, we have the reasonable result, by the division by zero calculus

$$x^2 + y^2 - (a^2 - b^2) = 0.$$

In the classical theory for quadratic curves, we have to arrange globally it by the division by zero calculus.

- The area $S(x)$ surrounded by two $x, y$ axises and the line passing a fixed point $(a, b), a, b > 0$ and a point $(x, 0)$ is given by

$$S(x) = \frac{bx^2}{2(x-a)}. \tag{9.12}$$

For $x = a$, we obtain, by the division by zero calculus, the very interesting value

$$S(a) = ab. \tag{9.13}$$

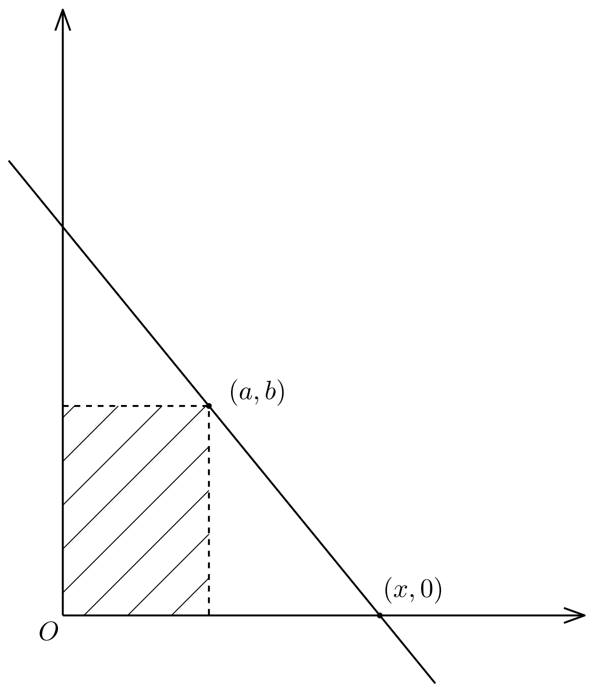


Figure 29.

- For example, for fixed point $(a, b)$; $a, b > 0$ and fixed a line $y = (\tan\theta)x, 0 < \theta < \pi$, we will consider the line $L(x)$ passing the two points $(a, b)$ and $(x, 0)$. Then, the area $S(x)$ of the triangle surround by the three lines $y = (\tan\theta)x$, $L(x)$ and the $x$ axis is given by

$$S(x) = \frac{b}{2}\frac{x^2}{x - (a - b\cot\theta)}.$$

For the case $x = a - b\cot\theta$, by the division by zero calculus, we have

$$S(a - b\cot\theta) = b(a - b\cot\theta).$$

Note that this is the area of the parallelogram through the origin and the point $(a, b)$ formed by the lines $y = (\tan\theta)x$ and the $x$ axis.

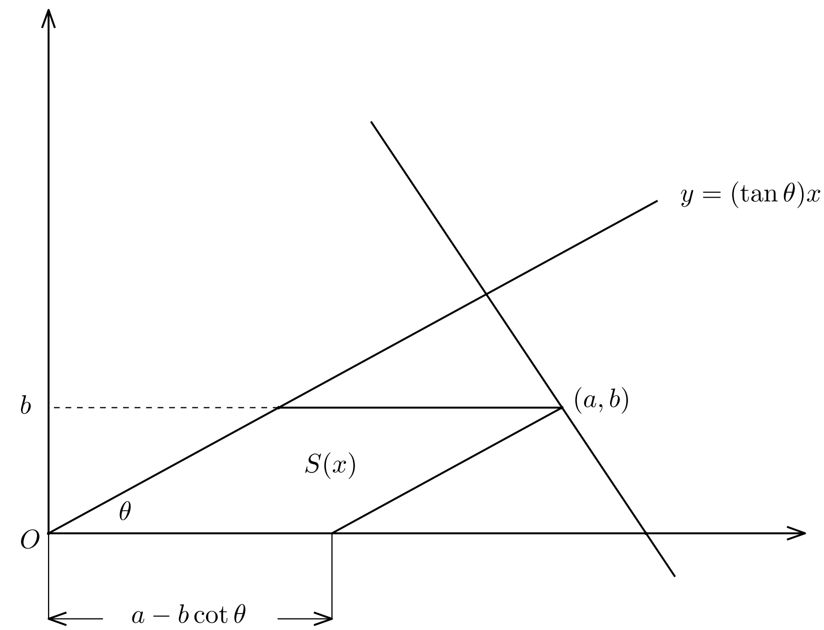

Figure 30.

- We consider an equilateral triangle with vertices $(\pm a/2, \sqrt{3}a/2)$ and the origin. The area $S(h)$ of the triangle surrounded by the three lines that the line through $(0, h + \sqrt{3}a/2)$ and $(-a/2, \sqrt{3}a/2)$, the line through $(0, h + \sqrt{3}a/2)$ and $(a/2, \sqrt{3}a/2)$ and the $x$- axis is given by .

$$S(h) = \frac{\left(h + (\sqrt{3}/2)a\right)^2}{2h}. \tag{9.14}$$

Then, by the division by zero calculus, we have, for $h = 0$,

$$S(0) = \frac{\sqrt{3}}{2}a^2.$$

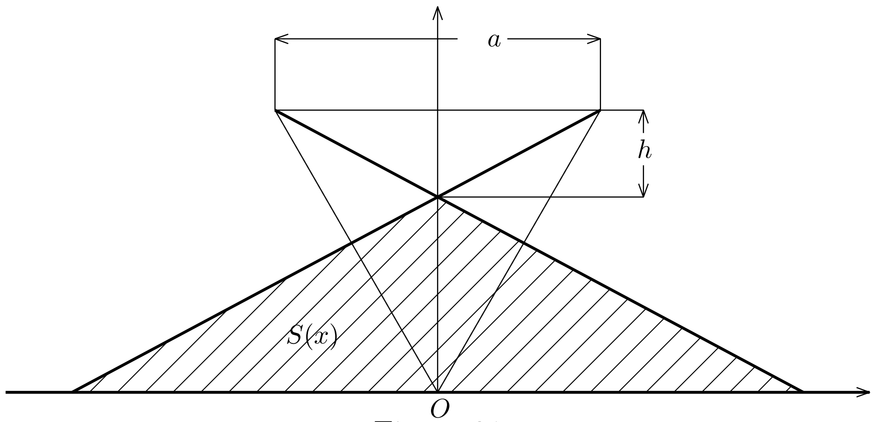

Figure 31.

- Similarly, we will consider the cone formed by the rotation of the line

$$\frac{kx}{a(k+h)} + \frac{y}{k+h} = 1$$

and the $x, y$ plane with center the $z$- axis ($a, h > 0$, and $a, h$ are fixed). Then, the volume $V(x)$ is given by

$$V(k) = \frac{\pi}{3}\frac{a^2(k+h)^3}{k^2}.$$

Then, by the division by calculus, we have the reasonable value

$$V(0) = \pi a^2 h.$$

- For example, for the plane equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \tag{9.15}$$

for $a = 0$, we can consider the line naturally, by the division by zero

$$\frac{y}{b} + \frac{z}{c} = 1. \tag{9.16}$$

- As in the line case, in the hyperbolic curve

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a, b > 0, \tag{9.17}$$

by the representations by parameters

$$x = \frac{a}{\cos\theta} = \frac{a}{2}\left(\frac{1}{t} + t\right)$$

and

$$y = \frac{b}{\tan\theta} = \frac{b}{2}\left(\frac{1}{t} - t\right)$$

the origin $(0,0)$ may be included as the point of the hyperbolic curve, as we see from the cases $\theta = \pi/2 = 0$ and $t = 0$.

In addition, from the fact, we will be able to understand that the asymptotic lines are the tangential lines of the hyperbolic curve.

The two tangential lines of (7.17) with gradient $m$ is given by

$$y = mx \pm \sqrt{a^2m^2 - b^2} \tag{9.18}$$

and the gradients of the asymptotic lines are

$$m = \pm\frac{b}{a}. \tag{9.19}$$

Then, we have asymptotic lines $y = \pm\frac{b}{a}x$ as tangential lines in (7.17).

The common points of (7.17) and (7.18) are given by

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2m}{\sqrt{a^2m^2 - b^2}}\right). \tag{9.20}$$

For the case $a^2m^2 - b^2 = 0$, we have they are $(0,0)$.

- We fix a circle

$$x^2 + (y-a)^2 = a^2, \quad a > 0. \tag{9.21}$$

At the point $(2a + d, 0), d > 0$, we consider two tangential lines for the circle. Let $2\theta$ is the angle between two tangential lines at the point $(2a+d, 0)$, Then, the area $S(h) = S(\theta)$ and the length $L(x) = L(\theta)$ are given by

$$S(h) = S(\theta) = \frac{a}{\sqrt{h}}(h + 2a)^{\frac{3}{2}} \tag{9.22}$$

$$= \frac{a^2}{\cos\theta}\left(\sin\theta + 2 + \frac{1}{\sin\theta}\right)$$

and

$$L(h) = L(\theta) = \frac{a}{\sqrt{h}}\sqrt{h+2a} \tag{9.23}$$
$$= a\left(\frac{1}{\cos\theta} + \tan\theta\right),$$

respectively. For $h = 0$ and $\theta = 0$, by diivision by zero calculus, we see that all are zero.

- We consider two spheres defined by

$$x^2 + y^2 + z^2 + 2a_j + 2b_j + 2c_j + 2d_j = 0, \quad j = 1, 2. \tag{9.24}$$

Then, the angle $\theta$ by two spheres is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2 - (d_1 + d_2)}{\sqrt{a_1^2 + b_1^2 + c_1^2 - 2d_1}\sqrt{a_2^2 + b_2^2 + c_2^2 - 2d_2}}. \tag{9.25}$$

If $\cos\theta = 0$, then, two spheres are orthogonal or one sphere is a point sphere.

- For the parabolic equation

$$y^2 = 4px,$$

two points $(pt^2, 2pt)$ and $(qt^2, 2qt)$ is a diameter is if and only if

$$(s-t)\{t(s+t)+2\} = 0; \quad s = -t - \frac{2}{t}$$

and the diameter $r$ is given by

$$r^2 = p^2(t-s)^2\{(t+s)^2 + 4\}.$$

Here, we should consider the case $t = s = 0$ as $r = 0$ and

$$0 = -0 - \frac{2}{0},$$

and the $x$ and $y$ axises are the orthogonal two tangential lines of the parabolic equation.

# 10 Applications to Wasan geometry

We will introduce typical applications of the division by zero calculus to Wasan geometry (traditional Japanese geometry), however, the results and their impacts will create some new fields in mathematics.

## 10.1 Circle and line

We will consider the fixed circle $x^2 + (y-b)^2 = b^2, b > 0$. For a touching circle with this circle and the $x$ axis is represented by

$$(x - 2\sqrt{ab})^2 + (y-a)^2 = a^2.$$

Then, we have

$$\frac{x^2+y^2}{\sqrt{a}} - 4\sqrt{b}x = 2\sqrt{a}(y-2b)$$

and

$$\frac{x^2+y^2}{a} - 4\sqrt{\frac{b}{a}}x = 2(y-2b).$$

Then, by the division by zero, we have the reasonable results the origin, that is the point circle of the origin, the $y$ axis and the line $y = 2b$. (H. Okumura: 2017.10.13.).

## 10.2 Three touching cirlces exteriously

For real numbers $z$, and $a, b > 0$, the point $(0, 2\sqrt{ab}/z)$ is denoted by $V_z$. H. Okumura and M. Watanabe gave the theorem in [13]:

**Theorem 7.** *The circle touching the circle $\alpha$: $(x-a)^2 + y^2 = a^2$ and the circle $\beta$: $(x+b)^2 + y^2 = b^2$ at points different from the origin $O$ and passing through $V_{z\pm1}$ is represented by*

$$\left(x - \frac{b-a}{z^2-1}\right)^2 + \left(y - \frac{2z\sqrt{ab}}{z^2-1}\right)^2 = \left(\frac{a+b}{z^2-1}\right)^2 \tag{10.1}$$

*for a real number $z \neq \pm 1$*

*The common external tangents of $\alpha$ and $\beta$ can be expressed by the equations*

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0. \tag{10.2}$$

Following our concept of the division by zero calculus, we will consider the case $z^2 = 1$ for the singular points in the general parametric representation of the touching circles.

### 10.2.1 Results

First, for $z = 1$ and $z = -1$, respectively by the division by zero calculus, we have from (10.1), surprisingly

$$x^2 + \frac{b-a}{2}x + y^2 \mp \sqrt{ab}y - ab = 0, \tag{10.3}$$

respectively [12].

Secondly, multiplying (10.1) by $(z^2 - 1)$, we immediately obtain surprisingly (10.2) for $z = 1$ and $z = -1$, respectively by the division by zero calculus.

In the usual way, when we consider the limiting $z \to \infty$ for (10.1), we obtain the trivial result of the point circle of the origin. However, the result may be obtained by the division by zero calculus at $w = 0$ by setting $w = 1/z$.

### 10.2.2 On the circle appeared

Let $\zeta$ be the circle expressed by (10.3) with minus sign. Then $\zeta$ meets the circles $\alpha$ in two points

$$P_a\left(2r_{\mathrm{A}}, 2r_{\mathrm{A}}\sqrt{\frac{a}{b}}\right), \quad Q_a\left(\frac{2ab}{9a+b}, -\frac{6a\sqrt{ab}}{9a+b}\right),$$

where $r_{\mathrm{A}} = ab/(a+b)$ (see Figure 31). Also it meets $\beta$ in points

$$P_b\left(-2r_{\mathrm{A}}, 2r_{\mathrm{A}}\sqrt{\frac{b}{a}}\right), \quad Q_b\left(\frac{-2ab}{a+9b}, -\frac{6b\sqrt{ab}}{a+9b}\right).$$

The line $P_aP_b$ is the external common tangent of the two circles $\alpha$ and $\beta$ on the upper half plane. The lines $P_aQ_a$ and $P_bQ_b$ intersect at the point $R : \left(0, -\sqrt{ab}\right)$, which lies on the remaining external common tangent of $\alpha$ and $\beta$. Furthermore, $\zeta$ is orthogonal to the circle with center $R$ passing through the origin.

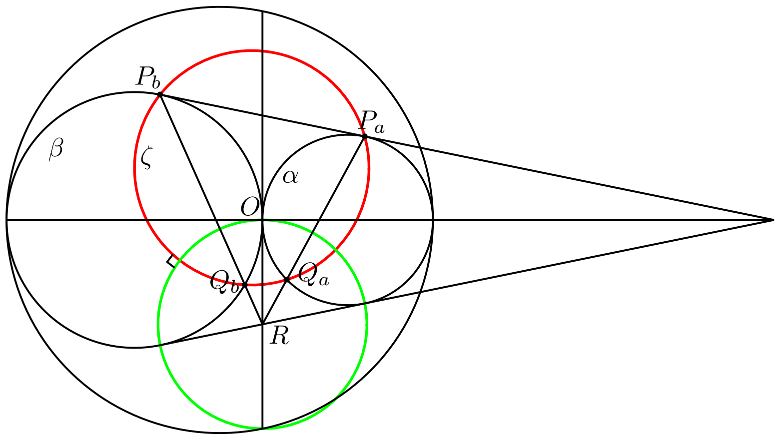


Figure 32.

## 10.3 The Descartes circle theorem

We recall the famous and beautiful theorem ([5, 24]):

**Theorem (Descartes)** *Let $C_i$ $(i = 1, 2, 3)$ be circles touching to each other of radii $r_i$. If a circle $C_4$ touches the three circles, then its radius $r_4$ is given by*

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}}. \tag{10.4}$$

As well-known, circles and lines may be looked as the same ones in complex analysis, in the sense of stereographic projection and many reasons. Therefore, we will consider whether the theorem is valid for line cases and point cases for circles. Here, we will discuss this problem clearly from the division by zero viewpoint. The Descartes circle theorem is valid except for one case for lines and points for the three circles and for one exception case, we can obtain very interesting results, by the division by zero calculus.

We would like to consider all the cases for the Descartes theorem for lines and point circles, step by step.

### 10.3.1 One line and two circles case

We consider the case in which the circle $C_3$ is one of the external common tangents of the circles $C_1$ and $C_2$. This is a typical case in this paper. We assume $r_1 \geq r_2$. We now have $r_3 = 0$ in (10.4). Hence

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{0} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 \cdot 0} + \frac{1}{0 \cdot r_1}} = \frac{1}{r_1} + \frac{1}{r_2} \pm 2\sqrt{\frac{1}{r_1 r_2}}.$$

This implies

$$\frac{1}{\sqrt{r_4}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

in the plus sign case. The circle $C_4$ is the incircle of the curvilinear triangle made by $C_1$, $C_2$ and $C_3$ (see Figure 32). In the minus sign case we have

$$\frac{1}{\sqrt{r_4}} = \frac{1}{\sqrt{r_2}} - \frac{1}{\sqrt{r_1}}.$$

In this case $C_2$ is the incircle of the curvilinear triangle made by the other three (see Figure 33).

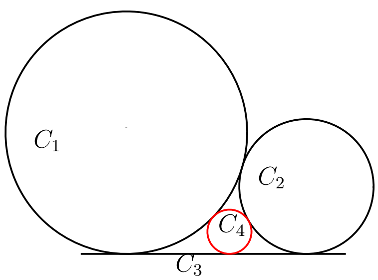

Figure 33.

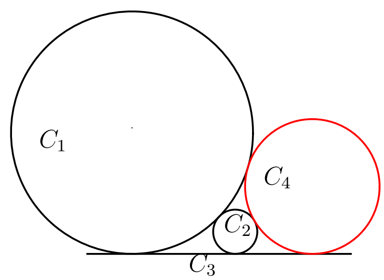

Figure 34.

Of course, the result is known. The result was also well-known in Wasan geometry [27] with the Descartes circle theorem itself.

### 10.3.2 Two lines and one circle case

In this case, the two lines have to be parallel, and so, this case is trivial, because then other two circles are the same size circles, by the division by zero $1/0 = 0$.

### 10.3.3 One point circle and two circles case

This case is another typical case for the theorem. Intuitively, for $r_3 = 0$, the circle $C_3$ is the common point of the circles $C_1$ and $C_2$. Then, there does not exist any touching circle of the three circles $C_j; j = 1, 2, 3$.

For the point circle $C_3$, we will consider it by limiting of circles attaching to the circles $C_1$ and $C_2$ to the common point. Then, we will examine the circles $C_4$ and the Descartes theorem.

In Theorem 7, by setting $z = 1/w$, we will consider the case $w = 0$; that is, the case $z = \infty$ in the classical sense; that is, the circle $C_3$ is reduced to the origin.

We look for the circles $C_4$ attaching with three circles $C_j; j = 1, 2, 3$. We set

$$C_4 : (x - x_4)^2 + (y - y_4)^2 = r_4^2. \tag{10.5}$$

Then, from the touching property we obtain:

$$x_4 = \frac{r_1 r_2 (r_2 - r_1) w^2}{D},$$

$$y_4 = \frac{2 r_1 r_2 \left(\sqrt{r_1 r_2} + (r_1 + r_2) w\right) w}{D}$$

and

$$r_4 = \frac{r_1 r_2 (r_1 + r_2) w^2}{D},$$

where

$$D = r_1 r_2 + 2\sqrt{r_1 r_2}(r_1 + r_2) w + (r_1^2 + r_1 r_2 + r_2^2) w^2.$$

By inserting these values to (10.5), we obtain

$$f_0 + f_1 w + f_2 w^2 = 0,$$

where

$$f_0 = r_1 r_2 (x^2 + y^2),$$

$$f_1 = 2\sqrt{r_1 r_2}\big((r_1 + r_2)(x^2 + y^2) - 2 r_1 r_2 y\big)$$

and

$$f_2 = (r_1^2 + r_1 r_2 + r_2^2)(x^2 + y^2) + 2 r_1 r_2 (r_2 - r_1) x - 4(r_1 + r_2) y + 4 r_1^2 r_2^2.$$

By using the division by zero calculus for $w = 0$, we obtain, for the first, for $w = 0$, the second by setting $w = 0$ after dividing by $w$ and for the third case, by setting $w = 0$ after dividing by $w^2$,

$$x^2 + y^2 = 0, \tag{10.6}$$

$$(r_1+r_2)(x^2+y^2)-2r_1r_2y=0 \tag{10.7}$$

and

$$(r_1^2+r_1r_2+r_2^2)(x^2+y^2)+2r_1r_2(r_2-r_1)x-4r_1r_2(r_1+r_2)y+4r_1^2r_2^2=0. \tag{10.8}$$

Note that (10.7) is the red circle in Figure 34 and its radius is

$$\frac{r_1r_2}{r_1+r_2} \tag{10.9}$$

and (10.8) is the green circle in Figure 34 whose radius is

$$\frac{r_1r_2(r_1+r_2)}{r_1^2+r_1r_2+r_2^2}.$$

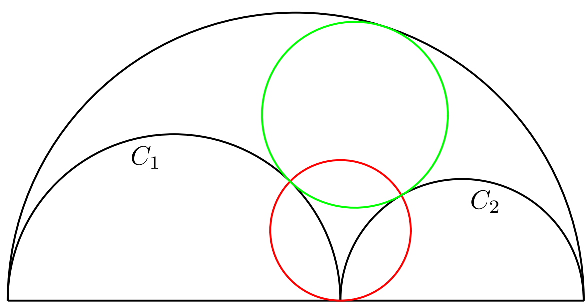

Figure 35.

When the circle $C_3$ is reduced to the origin, of course, the inscribed circle $C_4$ is reduced to the origin, then the Descartes theorem is not valid. However, by the division by zero calculus, then the origin of $C_4$ is changed suddenly for the cases (10.6), (10.7) and (10.8), and for the circle (10.7), the Descartes theorem is valid for $r_3=0$, surprisingly.

Indeed, in (9.4) we set $\xi=\sqrt{r_3}$, then (10.4) is as follows:

$$\frac{1}{r_4}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{\xi^2}\pm 2\frac{1}{\xi}\sqrt{\frac{\xi^2}{r_1r_2}+\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}.$$

and so, by the division by zero calculus at $\xi=0$, we have

$$\frac{1}{r_4}=\frac{1}{r_1}+\frac{1}{r_2}$$

which is (10.9). Note, in particular, that the division by zero calculus may be applied in many ways and so, for the results obtained should be examined some meanings. This circle (10.7) may be looked a circle touching the origin and two circles $C_1$ and $C_2$, because by the division by zero calculus

$$\tan\frac{\pi}{2} = 0,$$

that is a popular property.

Meanwhile, the circle (10.8) is the attaching circle with the circles $C_1$, $C_2$ and the beautiful circle with center $((r_2-r_1), 0)$ with radius $r_1+r_2$. The each of the areas surrounded by the three cicles $C_1$, $C_2$ and the circle of radius $r_1+r_2$ is called an arbelos, and the circle (10.7) is the famous Bankoff circle of the arbelos.

For $r_3 = -(r_1+r_2)$, from the Descartes identity (10.4), we have (10.4). That is, when we consider that the circle $C_3$ is changed to the circle with center $((r_2-r_1), 0)$ with radius $r_1+r_2$, the Descartes identity holds. Here, the minus sign shows that the circles $C_1$ and $C_2$ touch $C_3$ internally from the inside of $C_3$.

### 10.3.4 Two point circles and one circle case

This case is trivial, because, the exterior touching circle is coincident with one circle.

### 10.3.5 Three points case and three lines case

In these cases we have $r_j = 0, j = 1, 2, 3$ and the formula (10.4) shows that $r_4 = 0$. This statement is trivial in the general sense.

As the solution of the simplest equation

$$ax = b, \tag{10.10}$$

we have $x = 0$ for $a = 0, b \neq 0$ as the standard value, or the Moore-Penrose generalized inverse. This will mean in a sense, the solution does not exist; to solve the equation (10.10) is impossible. The zero will represent some **impossibility**.

In the Descartes theorem, three lines and three points cases, we can understand that the attaching circle does not exist, or it is the point and so the Descartes theorem is valid.

## 10.4 Circles and a chord

We recall the following result of the old Japanese geometry [26, 24, 13] (see Figure 35):

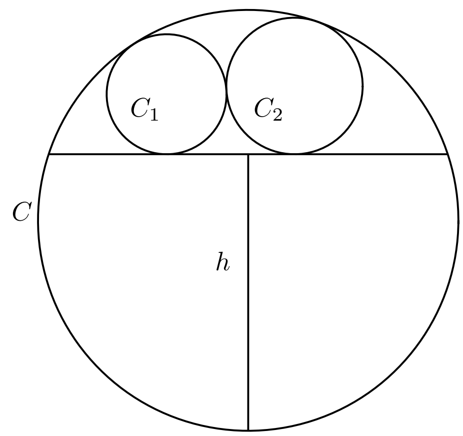


Figure 36.

**Lemma 10.** *Assume that the circle $C$ with radius $r$ is divided by a chord $t$ into two arcs and let $h$ be the distance from the midpoint of one of the arcs to $t$. If two externally touching circles $C_1$ and $C_2$ with radii $r_1$ and $r_2$ also touch the chord $t$ and the other arc of the circle $C$ internally, then $h$, $r$, $r_1$ and $r_2$ are related by*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{r_1 r_2 h}}.$$

We are interesting in the limit case $r_1 = 0$ or $r_2 = 0$.

### 10.4.1 Results

We introduce the coordinates in the following way: the bottom of the circle $C$ is the origin and tangential line at the origin of the circle $C$ is the $x$ axis and the $y$ axis is given as in the center of the circle $C$ is $(0, r)$. We denote the centers of the circles $C_j; j = 1, 2$ by $(x_j, y_j)$, then we have

$$y_1 = h + r_1, \quad y_2 = h + r_2.$$

Then, from the attaching conditions, we obtain the three equations:

$$(x_2 - x_1)^2 + (r_1 - r_2)^2 = (r_1 + r_2)^2,$$

$$x_1^2 + (h - r + r_1)^2 = (r - r_1)^2$$

and

$$x_2^2 + (h - r + r_2)^2 = (r - r_2)^2.$$

Solving the equations for $x_1$, $x_2$ and $r_2$, we get four sets of the solutions. Let $h = 2r_3$, $v = r - r_1 - r_3$. Then two sets are:

$$\begin{aligned} x_1 &= \pm 2\sqrt{r_3 v}, \\ x_2 &= \pm 2\frac{r_1\sqrt{rr_3} + r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} - \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

The other two sets are

$$\begin{aligned} x_1 &= \pm 2\sqrt{r_3 v}, \\ x_2 &= \mp 2\frac{r_1\sqrt{rr_3} - r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} + \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

We now consider the solution

$$\begin{aligned} x_1 &= 2\sqrt{r_3 v}, \\ x_2 &= 2\frac{r_1\sqrt{rr_3} + r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} - \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

Then

$$(x - x_2)^2 + (y - y_2)^2 - r_2^2 = \frac{g_0 + g_1 r_1 + g_2 r_1^2 + g_3}{(r_1 + r_3)^2},$$

where

$$g_0 = r_3^2(x^2 + y(y - 4r_3) + 4rr_3),$$

$$g_1 = 2r_3((x - \sqrt{rr_3})^2 + y^2 - (2r + 3r_3)y + 3rr_3),$$
$$g_2 = (x - 2\sqrt{rr_3})^2 + y^2 - 2r_3 y,$$

and
$$g_3 = 4r_3\sqrt{v}(r_1(\sqrt{r}y - \sqrt{r_3}x) - r_3\sqrt{r_3}x).$$

We now consider another solution

$$\begin{aligned} x_1 &= 2\sqrt{r_3 v}, \\ x_2 &= -2\frac{r_1\sqrt{rr_3} - r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} + \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

Then
$$(x - x_2)^2 + (y - y_2)^2 - r_2^2 = \frac{k_0 + k_1 r_1 + k_2 r_1^2 + k_3}{(r_1 + r_3)^2},$$

where
$$k_0 = r_3^2(x^2 + y(y - 4r_3) + 4rr_3),$$
$$k_1 = 2r_3((x + \sqrt{rr_3})^2 + y^2 - (2r + 3r_3)y + 3rr_3),$$
$$k_2 = (x + 2\sqrt{rr_3})^2 + y^2 - 2r_3 y,$$

and
$$k_3 = -4r_3\sqrt{v}(r_1(\sqrt{r}y + \sqrt{r_3}x) + r_3\sqrt{r_3}x).$$

We thus see that the circle $C_2$ is represented by

$$(g_0 + g_3) + g_1 r_1 + g_2 r_1^2 = 0$$

and
$$(k_0 + k_3) + k_1 r_1 + k_2 r_1^2 = 0.$$

For the symmetry, we consider only the above case. We obtain the division by zero calculus, first by setting $r_1 = 0$, the next by setting $r_1 = 0$ after dividing by $r_1$ and the last by setting $r_1 = 0$ after dividing by $r_1^2$,

$$g_0 + g_3 = 0,$$

$$g_1 = 0,$$

and

$$g_2 = 0.$$

That is,

$$\left(x - \sqrt{2rh - h^2}\right)^2 + (y - h)^2 = 0,$$

$$\left(x - \sqrt{\frac{rh}{2}}\right)^2 + \left(y - \left(r + \frac{3h}{4}\right)\right)^2 = r^2 + \frac{9}{16}h^2,$$

and

$$\left(x - \sqrt{2rh}\right)^2 + \left(y - \frac{h}{2}\right)^2 = \left(\frac{h}{2}\right)^2.$$

The first equation represents one $(\sqrt{2rh - h^2}, h)$ of the points of intersection of the circle $C$ and the chord $t$ (see Figure 36). The second equation expresses the red circle in the figure. The third equation expresses the circle touching $C$ externally, the $x$-axis and the extended chord $t$ denoted by the green circle in the figure. The last two circles are orthogonal to the circle with center origin passing through the points of intersection of $C$ and $t$.

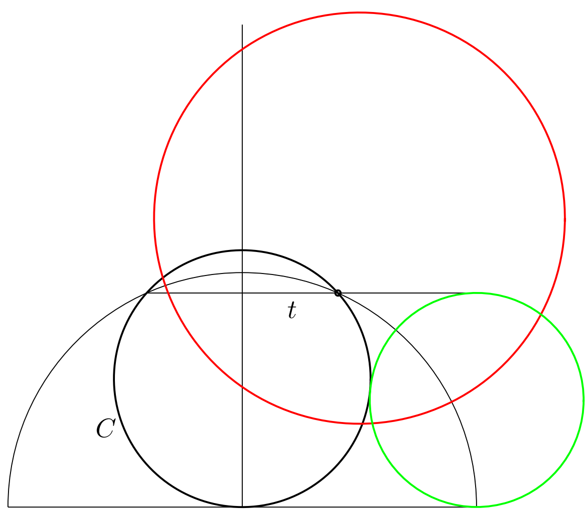


Figure 37

Now for the beautiful identity in the lemma, for $r_1 = 0$, we have, by the division by zero,

$$\frac{1}{0} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{0 \cdot r_2 h}}$$

and

$$r_2 = -\frac{h}{2}.$$

Here, the minus sigh will mean that the blue circle is attaching with the circle $C$ in the outside of the circle $C$; that is, we can consider that when the circle $C_1$ is reduced to the point $(\sqrt{2rh - h^2}, h)$, then the circle $C_2$ is suddenly changed to the blue circle and the beautiful identity is still valid. Note, in particular, the blue circle is attaching with the circle $C$ and the cord $t$.

Meanwhile, for the curious red circle, we do not know its property, however, we know curiously that it is orthogonal with the circle with the center at the origin and with radius $\sqrt{2rh}$ passing through the points $(\pm\sqrt{2rh - h^2}, h)$.

This subsection is based on the paper [17].

# 11 Conclusion

Apparently, the common sense on the division by zero with a long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on derivatives we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero.

This book is an elementary mathematics on our division by zero as the first publication of books for the topics. The contents have wide connections to various fields beyond mathematics. The authors expect the readers write some philosophy, papers and essays on the division by zero from this simple source book.

The representations of the contents are very important and delicate with delicate feelings to the division by zero with a long and mysterious history. Therefore, we hope the representations of the division by zero as follows:

- Various book publications by many native languages and with the author's idea and feelings.
- Some publications are like arts and some comic style books with pictures.
- Some T shirts design, some pictures, monument design may be considered.

The authors above may be expected to contribute to our culture and education. The topics will be interested in over 1000 millions people over the world on the world history.

For the people having the interest on the above projects, we will send our book source with many figure files.

The division by zero theory may be developed and expanded greatly.

We have to arrange globally our modern mathematics with our division by zero in our undergraduate level.

We have to change our basic ideas for our space and world.

We have to change globally our textbooks and scientific books on the division by zero.

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